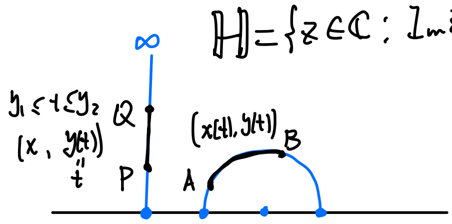
Hyperbolic Geometry : line = geodesic $<$ vertical line / half circle

$\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$



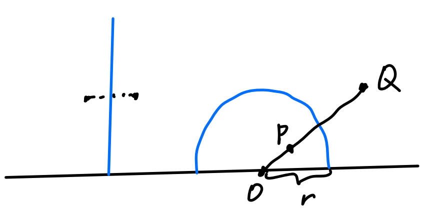
$$L(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)}\,dt$$

$$d(P,Q) = \int_{y_1}^{y_2} \underbrace{\frac{\sqrt{0^2+1^2}}{t}}_{= \frac{1}{t}}\,dt = \ln t\Big|_{y_1}^{y_2} = \ln y_2 - \ln y_1 = \ln \frac{y_2}{y_1}$$

· Isometry of $\mathbb{H}_{hyp.} = \{ f : \mathbb{H} \to \mathbb{H} ; d(f(P), f(Q)) = d(P,Q) \}$ for all $P, Q \in \mathbb{H}$

Isometries
- orientation reversing : reflection acrosses a geodesic $<$ reflection / inversion
- $\boxed{\text{orientation preserving}}$

$= \left\{ T_\gamma : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, ad - bc > 0, \; a, b, c, d \in \mathbb{R}. \right\}$



$|OP| \cdot |OQ| = r^2$

$$T_\gamma(z) = \frac{az+b}{cz+d} = \text{composition of even number of reflections}$$

$=$

$\{$ translations , rotations $\}$

translations: parabolic translation, hyperbolic translation.

- T_γ is a translation: if T_γ has no fixed point on $\mathbb{H}$

$\Rightarrow$ T_γ has fixed points on $\mathbb{RP}^1 = \mathbb{R}\cup\{\infty\}$

- 1 fixed pt. ← parabolic translation
- 2 fixed pts ← hyperbolic trans.

T_γ = composition of 2 reflections across two geodesics l_1, l_2 that do not intersect.

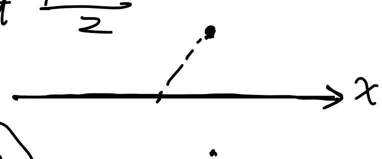
- T_γ is a rotation: if T_γ has a fixed point on $\mathbb{H}$.

Ex: $\gamma = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ $\quad T_\gamma(z) = \dfrac{1\cdot z - 1}{1\cdot z + 0} = \dfrac{z-1}{z}$

fixed point: $T_\gamma(z) = \dfrac{z-1}{z} = z \iff z - 1 = z^2$

$$z^2 - z + 1 = 0 \Rightarrow z = \frac{1 \pm \sqrt{(-1)^2 - 4}}{2} = \frac{1 \pm \sqrt{-3}}{2} = \frac{1 \pm \sqrt{3}\, i}{2}$$

$\Rightarrow T_\gamma$ is a rotation with fixed point $\frac{1+\sqrt{3}\, i}{2}$



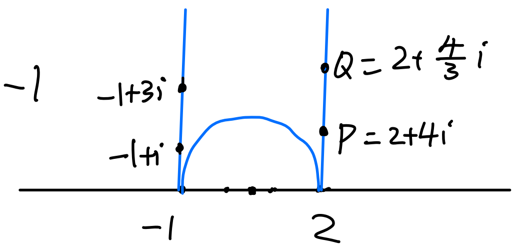
Ex: $\gamma = \begin{pmatrix} 1 & 2 \\ -1 & 2 \end{pmatrix}$ $\quad T_\gamma(z) = \dfrac{1\cdot z + 2}{-z + 2} = \dfrac{z+2}{-z+2}$

$\det \gamma = 2 - (-2) = 4 > 0.$

$$\frac{z+2}{-z+2} = z \iff z + 2 = z(-z+2) = -z^2 + 2z$$

$$\iff z^2 - z - 2 = 0 = (z-2)\cdot(z+1) \Rightarrow \boxed{z = -1, 2}$$

$P = 2+4i$, $\quad Q = \overset{2+\frac{4}{3}i}{\overset{\shortparallel}{\frac{6+4i}{3}}}$, $\quad i^2 = -1$



Isometry transforms geodesics to geodesics

$$T_\gamma P = \frac{2+4i+2}{-2-4i+2} = \frac{4+4i}{-4i} = -1+i$$

$$T_\gamma Q = \frac{\frac{6+4i}{3}+\overset{\frac{6}{3}}{\overset{\shortparallel}{2}}}{-\frac{6+4i}{3}+2=\frac{6}{3}} = \frac{12+4i}{-4i} = -1+3i$$

$$\frac{12+4i}{-4i} = \frac{12}{4i} - \frac{4i}{4i}$$

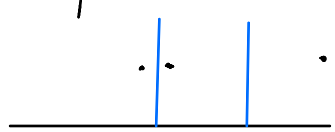
$$d(P,Q) = d(T_\gamma P, T_\gamma Q) = \ln\frac{3}{1} = \ln 3$$

$$\shortparallel$$

$$\ln\frac{4}{\frac{4}{3}} = \ln 3$$

Ex: $\quad z \mapsto z+a = \frac{1\cdot z + a}{0\cdot z + 1} = T_\gamma(z)$, $\quad \gamma = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$.

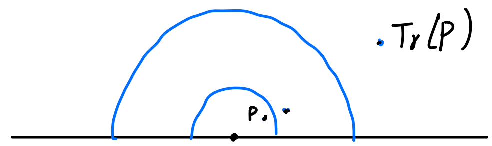
$T_\gamma(z) = z = z+a$ has a fixed point ∞



$\Rightarrow$ parabolic translation.

• $z \overset{\lambda>0}{\longmapsto} \lambda z = \dfrac{\lambda\cdot z+0}{0\cdot z+1}$

$\lambda z = z \overset{\lambda\neq 1}{\Longrightarrow} z = 0, \infty$ hyperbolic translation.



• Cross Ratio: $z_1, z_2, z_3, z_4 \in \mathbb{C}\cup\{\infty\}$

$$(z_1, z_2; z_3, z_4) = \frac{z_1-z_3}{z_1-z_4} \Big/ \frac{z_2-z_3}{z_2-z_4}$$

$$\boxed{T_\gamma(z) = \frac{a\cdot z+b}{cz+d}}.$$

$$= \frac{a(z+\frac{d}{c}) - \frac{ad}{c}+b}{cz+d}$$

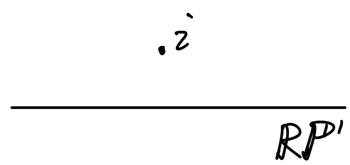
$$= \frac{a}{c^2} - \frac{ad-bc}{c^2}\cdot\frac{1}{z+\frac{d}{c}}$$

$$\boxed{\begin{gathered}(T_\gamma(z_1), T_\gamma(z_2); T_\gamma(z_3), T_\gamma(z_4)) \\ \parallel \\ (z_1, z_2; z_3, z_4)\end{gathered}}$$

$$\left(\frac{1}{z_1}, \frac{1}{z_2}; \frac{1}{z_3}, \frac{1}{z_4}\right) = (z_1, z_2; z_3, z_4)$$

$$= \frac{\frac{1}{z_1}-\frac{1}{z_3}}{\frac{1}{z_1}-\frac{1}{z_4}} \Big/ \frac{\frac{1}{z_2}-\frac{1}{z_3}}{\frac{1}{z_2}-\frac{1}{z_4}} = \frac{\frac{z_3-z_1}{\cancel{z_1z_3}}}{\frac{z_4-z_1}{\cancel{z_1z_4}}} \Big/ \frac{\frac{z_3-z_2}{\cancel{z_2z_3}}}{\frac{z_4-z_2}{\cancel{z_2z_4}}}$$

Ex: Find T_γ that sends $\boxed{i \mapsto i}$
$\gamma \in SL(2,\mathbb{R})$ $\quad \infty \mapsto 3$
$\quad 0 \mapsto -\frac{1}{3}$



$$\left(z, i; \infty, 0\right) = \left(\underbrace{T_\gamma(z)}_{w}, \underbrace{i}_{T_\gamma(i)}; \underbrace{3}_{T_\gamma(\infty)}, \underbrace{-\tfrac{1}{3}}_{T_\gamma(0)}\right)$$

$$= \frac{\frac{z-\infty}{z-0}}{\frac{i-\infty}{i-0}} = \frac{i}{z}\cdot\underbrace{\frac{z-\infty}{i-\infty}}_{=1}$$

$$= \frac{w-3}{w+\frac{1}{3}} \Big/ \frac{i-3}{i+\frac{1}{3}} = \frac{w-3}{w+\frac{1}{3}}\cdot\frac{i+\frac{1}{3}}{i-3}$$

$$\frac{i+\frac{1}{3}}{i-3} = \frac{1}{3}\cdot\frac{(3i+1)}{-3+i} = \frac{(\frac{1}{3}+i)\cdot(-3-i)}{(-3+i)\cdot(-3-i)} = \frac{(-1+1)-\frac{10}{3}i}{10} = -\frac{i}{3}$$

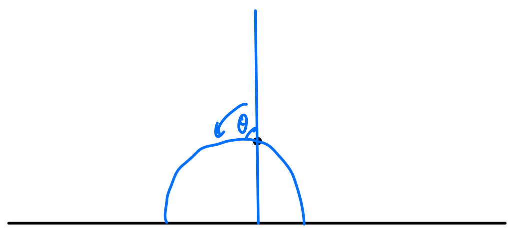
$$\frac{\cancel{i}}{z} = \frac{w-3}{w+\frac{1}{3}}\cdot\left(-\frac{\cancel{i}}{3}\right) = -\frac{w-3}{3w+1}$$

$$\Rightarrow -(3w+1) = z\cdot(w-3) = zw-3z$$
$$-(3w+1) = -3w-1$$

$$\Rightarrow -1+3z = zw+3w = (z+3)w \Rightarrow \boxed{T_\gamma(z) = w = \frac{3z-1}{z+3}}$$

$$T_\gamma = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix} = T_{\begin{pmatrix} \frac{3}{2} & \frac{-1}{2} \\ \frac{1}{2} & \frac{3}{2} \end{pmatrix}}.$$

$$i \mapsto \frac{3i-1}{i+3} = i, \quad \infty \mapsto \frac{3\cdot\infty-1}{\infty+3} = 3, \quad 0 \mapsto -\frac{1}{3}$$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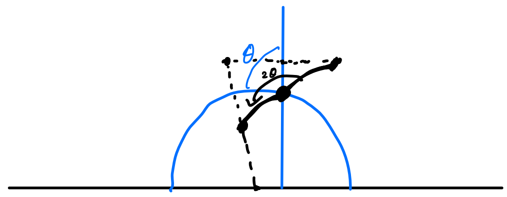
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