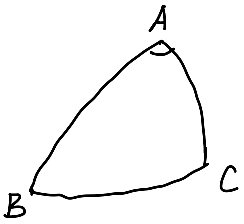
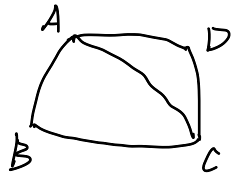
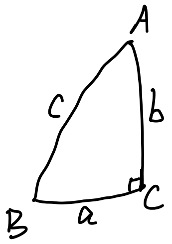
Spherical Geometry:



$$|\triangle ABC| = A+B+C-\pi$$

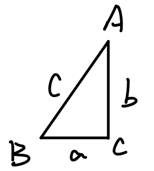


$$\text{Area} = A+B+C-2\pi$$



$$\sin A = \frac{\overset{a\text{ small}}{\overset{\sim a}{\boxed{\sin a}}}}{\underset{c\text{ small}\ c}{\boxed{\sin c}}}, \quad \cos A = \frac{\overset{\sim b}{\boxed{\sin b}}}{\underset{\sim c}{\boxed{\sin c}}} \cdot \boxed{\cos a} \sim 1$$

$$\downarrow \qquad\qquad \downarrow$$



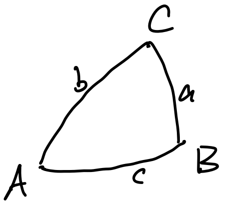
$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \overset{x\text{ small}}{\sim} x$$

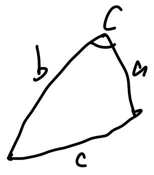
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots \overset{x\text{ small}}{\sim} 1 - \frac{x^2}{2}$$

. Law of sine:

$$\frac{\overset{\sim a}{\boxed{\sin a}}}{\sin A} = \frac{\overset{\sim b}{\boxed{\sin b}}}{\sin B} = \frac{\overset{\sim c}{\boxed{\sin c}}}{\sin C}$$



Law of cosine:
for sides

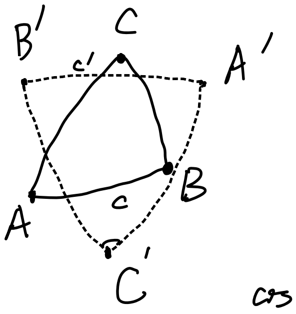


$$\boxed{\cos c} \overset{+}{=} \boxed{\cos a} \cdot \boxed{\cos b} + \boxed{\sin a} \cdot \boxed{\sin b} \cdot \cos C$$

$$\begin{array}{ccccc} 2 & 2 & 2 & 2 & 2 \\ 1-\frac{c^2}{2} & \left(1-\frac{a^2}{2}\right) & \left(1-\frac{b^2}{2}\right) & a & b \\ & \multicolumn{2}{c}{\|} & & \\ & \multicolumn{2}{c}{1-\frac{1}{2}(a^2+b^2)} & & \end{array}$$

$$(1 = 1)$$

$$\rightsquigarrow \quad c^2 = a^2+b^2-2\cdot ab\cdot\cos C$$



$$\boxed{\cos\angle C = \ominus\cos A\cdot\cos B+\sin A\cdot\sin B\cdot\cos c}$$

$$\angle C = \pi - c', \quad \angle A = \pi - a', \quad \angle B = \pi - b'$$

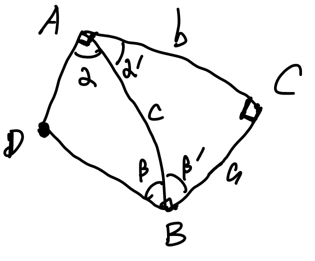
$$\cos\angle C = \cos(\pi - c') = -\cos C' = -\Big(\underset{\underset{-\cos\angle A}{\|}}{\underline{\cos a'}}\cdot\underset{\underset{-\cos\angle B}{\|}}{\underline{\cos b'}}+\underset{\underset{\sin\angle A}{\|}}{\underline{\sin a'}}\cdot\underset{\underset{\sin\angle B}{\|}}{\underline{\sin b'}}\,\cos\underset{\underset{(-\cos C)}{\|}}{\underline{\overset{\overset{\pi - c}{\|}}{C'}}}\Big)$$

SSS ⇒ congruence

AAA ⇗

↑ not true in Euclidean

Ex 10.27



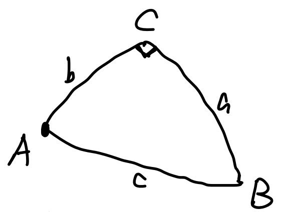
$$\boxed{\cos D = \sin a\cdot\sin b}$$

$$\underline{\sin\alpha'} = \sin\left(\frac{\pi}{2}-\alpha\right) = \underline{\cos\alpha}$$
$$\cos\alpha' = \cos\left(\frac{\pi}{2}-\alpha\right) = \sin\alpha$$

$$\cos D = -\cos\alpha\cdot\cos\beta + \sin\alpha\cdot\sin\beta\cdot\cos c \leftarrow \text{side}$$
$$= -\underline{\sin\alpha'}\,\sin\beta' + \underline{(+\cos\alpha')}\cdot(+\cos\beta')\cdot\cos C$$
$$= \underline{-\frac{\sin a}{\sin C}\cdot\frac{\sin b}{\sin C}} + \underline{\frac{\sin b}{\sin C}\cdot\underline{\cos a}\cdot\frac{\sin a}{\sin C}\cdot\underline{\cos b}\cdot\underline{\cos C}}$$

$$= -\frac{\sin a\cdot\sin b}{\sin^2 C} + \frac{\sin a\cdot\sin b}{\sin^2 C}\underset{\underline{\cos C}}{\underset{\|}{\boxed{\cos a\cdot\cos b}}}\cdot\underline{\cos C}\,.$$

$$= -\frac{\sin a\cdot\sin b}{\cancel{\sin^2 C}}\cdot\Big(\underset{\cancel{\sin^2 C}}{\underset{\|}{\underline{1-\cos^2 C}}}\Big) = -\sin a\cdot\sin b\,.$$



$$\text{Aea } \triangle = A+B+\overset{\frac{\pi}{2}}{\overset{\|}{C}}-\pi = A+B-\frac{\pi}{2}$$

$$\sin\triangle = \sin\left(A+B-\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2}-(A+B)\right)$$
$$= -\cos(A+B)$$

$$\left(\cos A = \frac{\sin b}{\sin C}\cdot\cos a\right)$$

$$\sin\triangle = -\cos(A+B) = -\left(\cos A\cdot\cos B - \sin A\cdot\sin B\right)$$

$$= \ominus\underset{\|}{\boxed{\cos A}}\cdot\cos B + \sin A\cdot\sin B$$

$$-\frac{\sin b}{\sin C}\cdot\cos a\cdot\frac{\sin a}{\sin C}\cos b + \frac{\sin a}{\sin C}\cdot\frac{\sin b}{\sin C} = -\frac{\sin a\cdot\sin b}{\sin^2 C}\cos a\cdot\cos b + \frac{\sin a\cdot\sin b}{\sin^2 C}$$

$$= \frac{\sin a\cdot\sin b}{\sin^2 C}\cdot(-\cos a\cdot\cos b+1) = \frac{\sin a\cdot\sin b}{1-\cos^2 C}(1-\cos C).$$

$$= \frac{\sin a\cdot\sin b}{(1+\cos C)\cdot(\cancel{1-\cos C})}\cdot(\cancel{1-\cos C}) = \frac{\sin a\cdot\sin b}{1+\cos C} = \frac{\sin a\cdot\sin b}{1+\cos a\cdot\cos b}$$

Law of cosine for sides: $\cos c = \cos a\cdot\cos b + \sin a\cdot\sin b\cdot\cos C$.

If $C=\frac{\pi}{2}$, $\cos c = \cos a\cdot\cos b$ (Pythagorean Thm)

$$\frac{\cos a+\cos b}{1+\cos C} = \cos\Delta = \cos\left(A+B-\frac{\pi}{2}\right) = \sin(A+B) = \sin A\cdot\underline{\cos B} + \cos A\cdot\sin B$$

$$= \frac{\sin a}{\sin C}\cdot\frac{\sin a}{\sin C}\cdot\cos b + \frac{\sin b}{\sin C}\cos a\cdot\frac{\sin b}{\sin C}$$

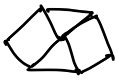
$$= \frac{\overset{1-\cos^2 a}{\overset{\shortparallel}{\boxed{\sin^2 a}}}\cdot\cos b}{\sin^2 C} + \frac{\sin^2 b\cdot\cos a}{\sin^2 C}$$

$$= \frac{\cos b}{\sin^2 C} - \frac{\boxed{\cos^2 a\cdot\cos b}}{\sin^2 C} = \underline{\cos a\cdot\cos C} + \frac{\cos a}{\sin^2 C} - \frac{\cos b\cdot\cos C}{\sin^2 C}$$

$$= \frac{1}{\sin^2 C}\left(\cos b+\cos a-\cos C\cdot\underline{(\cos a+\cos b)}\right)$$

$$= \frac{(\cos a+\cos b)\cdot}{\underset{\underset{(1+\cos C)\cdot(1-\cos C)}{\shortparallel}}{\boxed{1-\cos^2 C}}}(1-\cos C) = \frac{\cos a+\cos b}{1+\cos C}.$$

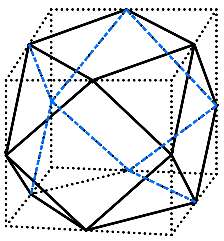
<u>10.37</u>: (3, 4, 3, 4)



cuboctahedron:

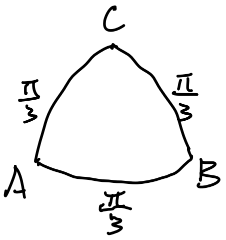
6 squares

8 triangles



Projecting to the sphere to get a semiregular tiling. A great circle is divided into 6 arcs of equal length.

$\Rightarrow$ side length $= \frac{2\pi}{6} = \frac{\pi}{3}$.

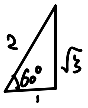


Law of Cosine for sides:

$$\cos\frac{\pi}{3} = \cos\frac{\pi}{3}\cdot\cos\frac{\pi}{3} + \sin\frac{\pi}{3}\cdot\sin\frac{\pi}{3}\cdot\cos C$$

$$\shortparallel \qquad\qquad \shortparallel$$

$$\frac{1}{2} \qquad \frac{1}{2}\cdot\frac{1}{2} + \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{3}}{2}\cdot\cos C$$



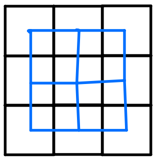
$$\Rightarrow \quad \cos C = \frac{\frac{1}{2}-\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \Rightarrow C = \arccos\left(\frac{1}{3}\right).$$

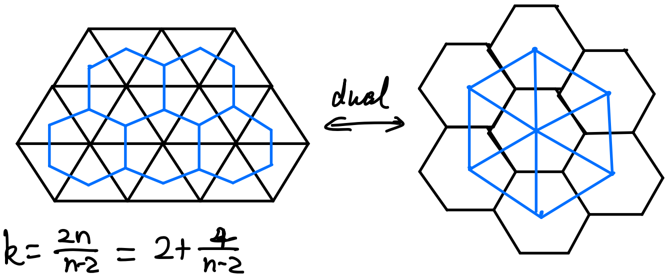
$\Rightarrow$ percentage of triangles: $\frac{(3C-\pi)8}{4\pi} = \frac{6\arccos(\frac{1}{3})}{\pi} - 2 \sim 0.35$

squares: $3 - \frac{6\arccos(\frac{1}{3})}{\pi} \sim 0.65$

Tiling on the plane

regular:





$$\frac{n-2}{n}\pi k = 2\pi \Leftrightarrow \quad k = \frac{2n}{n-2} = 2 + \frac{4}{n-2}$$

$n, k \geq 3$: integer

$n=3 \Rightarrow k=6$

$n=4 \Rightarrow k=4$

$n=5 \Rightarrow k=\frac{10}{3}$ X

$n=6 \Rightarrow k=3$ $\qquad n \geq 7 \Rightarrow k<3$

semiregular: 8 types

5 tiles at a vertex: (3 3 3 3 6), (3,3,3,4,4), (3,3,4,3,4)

4 tiles - - - : (3,4,6,4) (3,6,3,6)

3 tiles - - - : (3,12,12), (4,6,12), (4,8,8)

↑

$$\frac{l-2}{l}\pi + \frac{m-2}{m}\pi + \frac{n-2}{n}\pi = 2\pi \Rightarrow \frac{2}{l} + \frac{2}{m} + \frac{2}{n} = 1$$

If one of (l,m,n) is odd, then the other 2 numbers coincide.

For example, (3,7,42), (3,8,24) do not occur.