$\mathbb{H} = \{(x,y) : y>0\}$ $\qquad ds^2 = \dfrac{dx^2+dy^2}{y^2}$ $\qquad L(\gamma) = \int_a^b \dfrac{\sqrt{x'^2+y'^2}}{y}\,dt$

hyperbolic metric

$$\mathrm{Isom}(\mathbb{H}) = \{f : \mathbb{H} \to \mathbb{H},\ f^*ds^2 = ds^2\}$$

$$\cup$$

$$\mathrm{Isom}^+(\mathbb{H}) = \{f \in \mathrm{Isom}(\mathbb{H}),\ \underline{f \text{ preserves orientation}}\}$$

$$= \{T_\gamma : \gamma \in SL(2,\mathbb{R})\}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad a,b,c,d \in \mathbb{R},\ ad-bc=1.$$

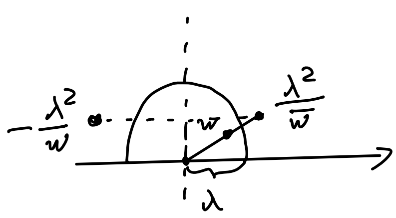
$$T_\gamma(z) = \frac{az+b}{cz+d} = \begin{cases} \dfrac{az+b}{d} = \dfrac{a}{d}z + \dfrac{b}{d} & (c=0) \quad \text{translation} \circ \text{rescaling} \\[2ex] \dfrac{a}{c} - \dfrac{\overbrace{ad-bc}^{=1}}{c^2\left(z+\frac{d}{c}\right)} & (c \neq 0) \end{cases}$$

$$z \xmapsto{\text{translation}} z + \frac{d}{c} \longrightarrow -\frac{1}{c^2\left(z+\frac{d}{c}\right)} \xrightarrow{\text{translation}} \frac{a}{c} - \frac{1}{c^2\left(z+\frac{d}{c}\right)}$$

$$\parallel$$

$$w \longmapsto -\frac{\lambda^2}{w}$$

inversion $\quad \dfrac{\lambda^2}{\bar{w}} \quad$ reflection

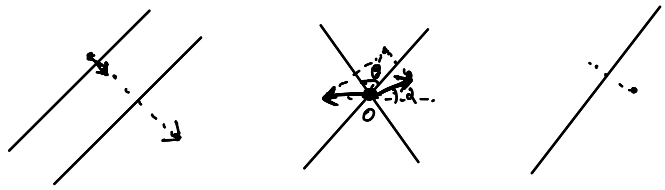


Thm: Any isometry of $\mathbb{H}$ is composition of reflections.

Thm: Any isometry of $\mathbb{R}^2$ is composition of reflections.

Euclidean plane

translation, rotation, reflection, glide reflection



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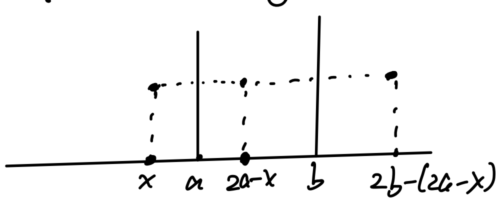
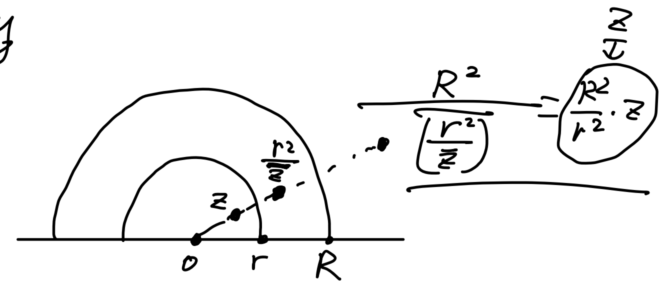
reflections in hyperbolic geometry



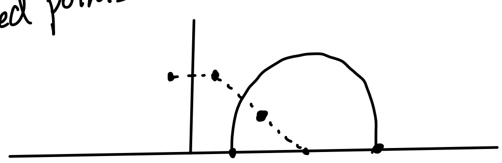
$= 2(b-a)+x$

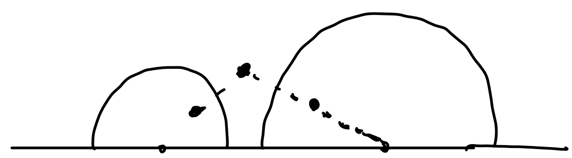


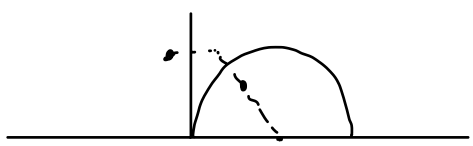
$$\frac{R^2}{\overline{\left(\frac{r^2}{\bar{z}}\right)}} = \frac{R^2}{r^2}\cdot z$$

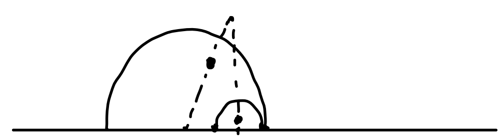
translations

no fixed points



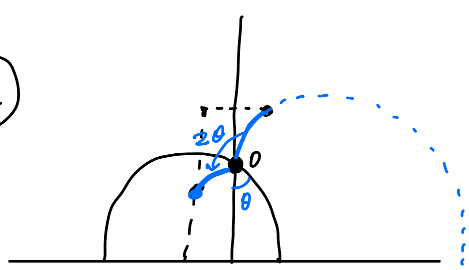


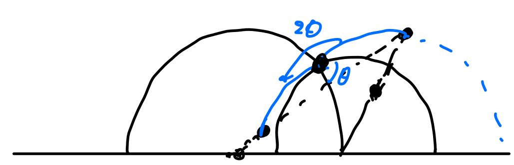


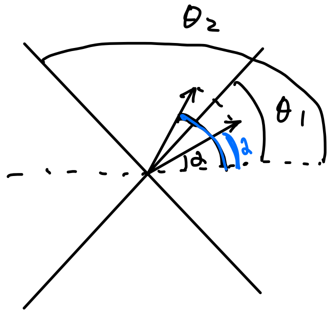


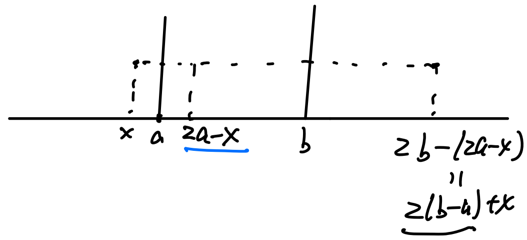
1 fixed point.

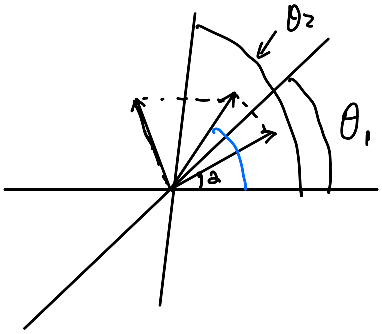
rotation











$$2(\theta_1-\alpha)+\alpha = 2\theta_1-2\alpha+\alpha = 2\theta_1-\alpha$$

$$\Downarrow$$

$$2\theta_2-(2\theta_1-\alpha) = 2(\theta_2-\theta_1)+\alpha$$

- $\mathrm{Isom}^+(\mathbb{H}) = \left\{ T_\gamma : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R}) \right\}$

$ad-bc=1.$

$$\frac{az+b}{cz+d} = z \iff az+b = (cz+d)z = cz^2+dz$$

$$\iff c\cdot z^2+(d-a)z-b=0$$

$$z = \frac{-c \pm \sqrt{(d-a)^2-4c(-b)}}{2c} = \frac{-(d-a)\pm\sqrt{(d-a)^2+4bc}}{2c}$$

T_γ has a fixed point in $\mathbb{H} \iff \boxed{(d-a)^2+4bc<0}$

$\Updownarrow$ $\qquad \mathbb{H} = \{ z : y>0 \}$, $z = x+iy$, $y = \mathrm{Im}(z)$

T_γ is a rotation

$(d-a)^2 + 4bc > 0 \implies T_\gamma$ has no fixed point in $\mathbb{H}$

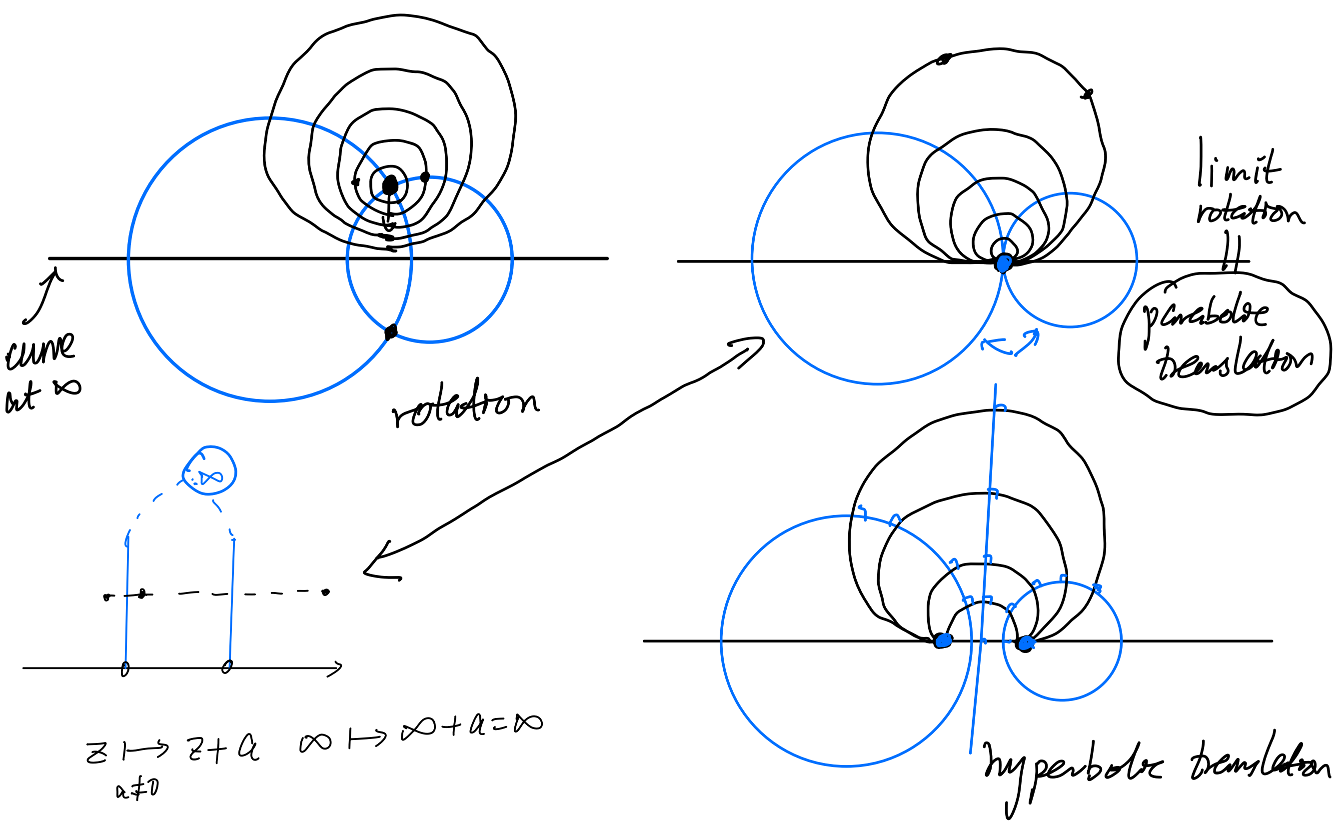
T_γ has 2 fixed points on $\mathbb{R} \cup \{\infty\}$

$\implies T_\gamma$ is translation (hyperbolic translation)

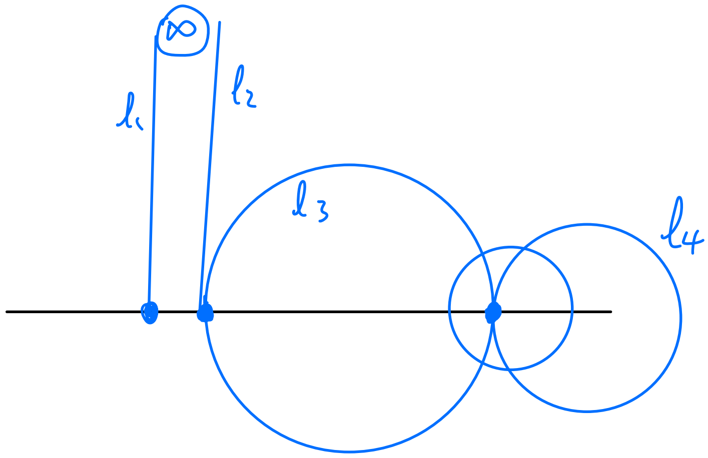
$(d-a)^2 + 4bc = 0 \implies T_\gamma$ has no fixed point in $\mathbb{H}$

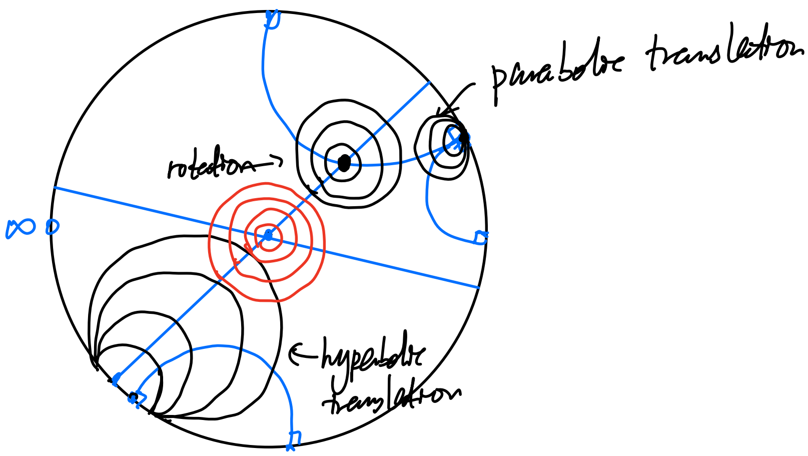
T_γ has 1 fixed point on $\mathbb{R} \cup \{\infty\}$

T_γ is translation (parabolic translation)



$z \mapsto z + a$, $a \neq 0$ $\quad \infty \mapsto \infty + a = \infty$





Poincaré unit disk