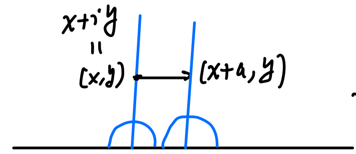
Hyperbolic geometry : Upper half plane.

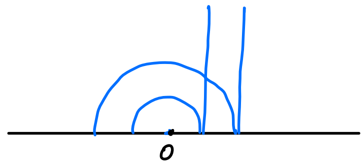
Examples of Isometries :

· translation



$\underline{z \mapsto z+a}$

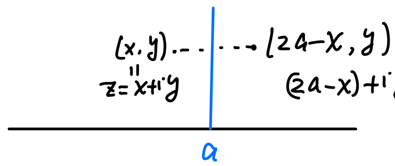
· scaling



$\underline{z \mapsto \lambda \cdot z \quad \lambda>0}$.

· reflection

$z \mapsto 2a-\bar{z}$



$2a-\bar{z}$

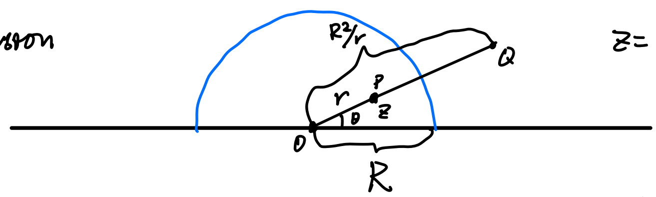
$\parallel$

$(2a-x)+iy = 2a-(x-iy)$

$\bar{z}=x-iy$

$z=x+iy$.

· inversion



$z = r\cdot e^{i\theta} = r\cdot(\cos\theta + i\cdot\sin\theta)$

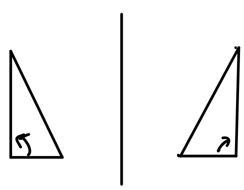
$= (r\cdot\cos\theta + i\cdot r\sin\theta)$

$\bar{z} = r\cdot e^{-i\theta}$

$$|0P|\cdot|0Q| = R^2 \Rightarrow |0Q| = \frac{R^2}{|0P|} = \frac{R^2}{r}$$

$$Q \Leftrightarrow |0Q|\cdot e^{i\theta} = \frac{R^2}{r}\cdot e^{i\theta} = \frac{R^2}{r\cdot e^{-i\theta}} = \frac{R^2}{\bar{z}}$$

$$\boxed{z \mapsto \frac{R^2}{\bar{z}}}$$



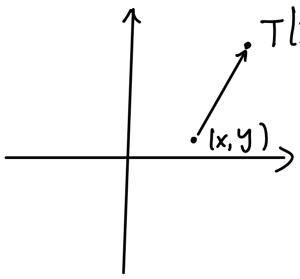
Isometry
- orientation preserving (translation, rescaling, ...)
- orientation reversing (reflection, inversion, ...)

Euclidean:

translation, rotation ← orientation preserving

reflection, glide reflection ← orientation reversing.

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \xmapsto{T} \underline{A\cdot v + w_0} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$



$$A^T \cdot A = I_2 = A \cdot A^T \quad (A^T = A^{-1})$$

orthogonal matrix.

$$\Downarrow$$

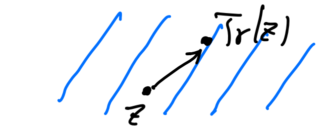
$$\det(A) = ad - bc = 1 \;<\; \begin{array}{l} A = I_2 \\ A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \rightsquigarrow \text{rotation} \end{array}$$

$$\text{or} = -1 \;<\; \begin{array}{l} \text{reflection} \\ \text{glide reflection} \end{array}$$

· Hyperbolic isometry. linear fractional transformation

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad \underline{ad - bc = 1 . = \det(\gamma)} \qquad \underline{a, b, c, d \in \mathbb{R}.}$$

$$\rightsquigarrow T_\gamma(\underset{x+iy}{\underset{\uparrow}{z}}) = \frac{az+b}{cz+d}$$



· $c = 0$, $T_\gamma(z) = \dfrac{az+b}{0\cdot z + d} = \dfrac{az+b}{d} = \underline{\left(\dfrac{a}{d}\right) z + \left(\dfrac{b}{d}\right)}$ is an isometry.

$1 = ad - bc = \underline{ad} > 0 \Rightarrow a, d$ same sign

· $c \neq 0$, $\underset{T_\gamma(z)}{\underset{\parallel}{\dfrac{az+b}{cz+d}}} = \dfrac{a\left(z + \frac{d}{c}\right) - \frac{ad}{c} + b}{c\left(z + \frac{d}{c}\right)} = \dfrac{a}{c} - \dfrac{ad - bc}{c^2\left(z + \frac{d}{c}\right)}$

$$\underset{\text{translation}}{z \to z + \frac{d}{c}} \underset{\text{reflection + inversion}}{\longrightarrow} -\frac{\lambda}{z + \frac{d}{c}} \underset{\text{translation}}{\longrightarrow} \frac{a}{c} - \frac{\lambda}{\left(z + \frac{d}{c}\right)^2}$$

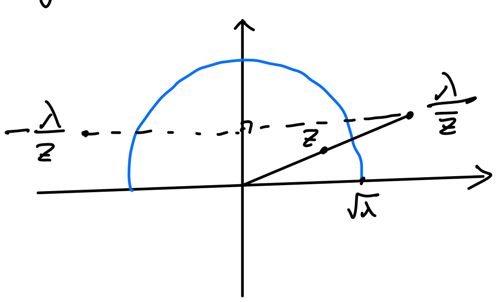
$$\lambda = \frac{ad - bc}{c^2} = \frac{1}{c^2} > 0$$

inversion: reflection w.r.t. a circle.

$z \mapsto -\frac{\lambda}{z}$

reflection w.r.t. circle of radius $\sqrt{\lambda}$: $\frac{\lambda}{\bar z}$

$2\cdot 0 - \overline{\left(\frac{\lambda}{\bar z}\right)}$ reflection w.r.t. $x=0$.



reflection · $z \mapsto \frac{R^2}{\bar z}$

· $z \mapsto 2a - \bar z$

$$-\frac{\lambda}{z} = \frac{0\cdot z - \sqrt{\lambda}}{\frac{1}{\sqrt{\lambda}}\cdot z + 0} = T_{\begin{pmatrix} 0 & -\sqrt{\lambda} \\ \frac{1}{\sqrt{\lambda}} & 0 \end{pmatrix}}(z)$$

$$T_{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}(z) = \frac{az+b}{cz+d}$$

linear fractional transformation

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\det(\gamma) = ad - bc > 0$. $\rightsquigarrow \frac{1}{\sqrt{\det(\gamma)}}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$T_\gamma(z) = \frac{az+b}{cz+d} = \frac{(a\cdot t)\cdot z + b\cdot t}{(c\cdot t)z + d\cdot t} = T_{\begin{pmatrix} ta & tb \\ tc & td \end{pmatrix}}(z)$$

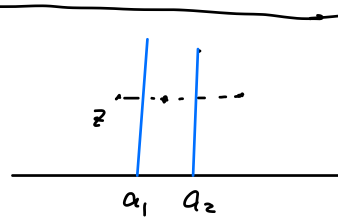
$$\det(\underline{t\cdot\gamma}) = t^2\det(\gamma)$$

$$T_\gamma(z) = T_{\frac{1}{\sqrt{\det(\gamma)}}\gamma}(z)$$

$\left(\det\left(\frac{\gamma}{\sqrt{\det(\gamma)}}\right) = \left(\frac{1}{\sqrt{\det(\gamma)}}\right)^2 \cdot \det(\gamma) = 1.\right)$

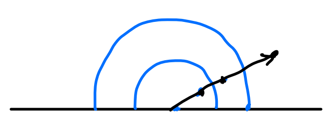
· translation $\quad z \mapsto z + a$

$$2a_1 - \bar z \mapsto 2a_2 - \overline{(2a_1 - \bar z)} = 2(a_2 - a_1) + z$$



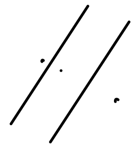
· scaling $\quad z \mapsto \lambda z$

$$\frac{R_1^2}{\bar z} \mapsto \frac{R_2^2}{\overline{\left(\frac{R_1^2}{\bar z}\right)}} = z\cdot\frac{R_2^2}{R_1^2}$$

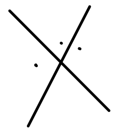


Euclidean

translation

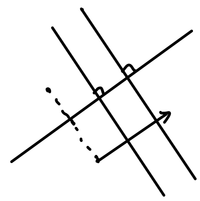


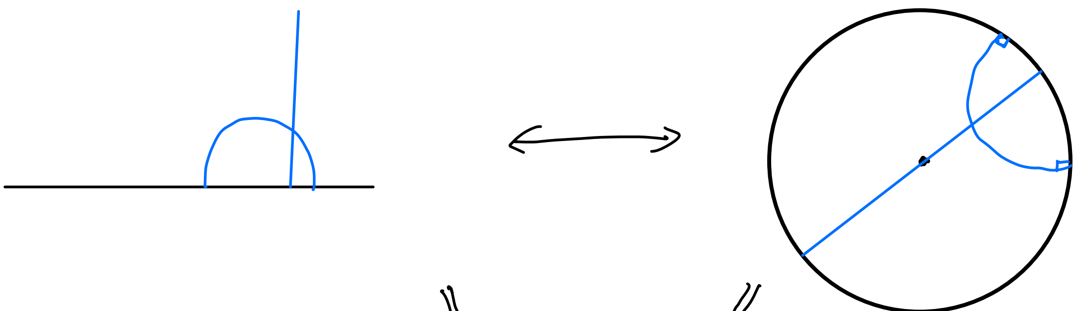
rotation

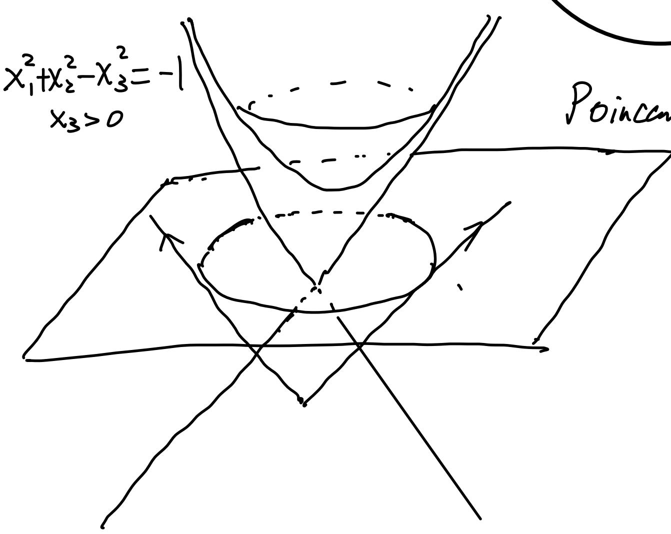


reflection

glide reflection







$$\mathrm{Isom}^+(\mathbb{H}) = \left\{ T_\gamma(z) : \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix},\ ad-bc=1 \right\}$$

upper half plane

$\left\{ \text{isometries of } \mathbb{H} \text{ that preserve orientation} \right\}$

$$\gamma_1 \cdot \gamma_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma$$

$$T_{\gamma_1} \circ T_{\gamma_2} \quad = \quad T_\gamma$$

$$T_{\gamma_1} \circ \underline{T_{\gamma_2}(z)} = T_{\gamma_1}\left(\left(\frac{a_2 z + b_2}{c_2 z + d_2}\right)\right) = \frac{a_1 \cdot \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \cdot \frac{a_2 z + b_2}{c_2 z + d_2} + d_1}$$

$$= \frac{a_1(a_2 z + b_2) + b_1(c_2 z + d_2)}{c_1(a_2 z + b_2) + d_1(c_2 z + d_2)} = \frac{(a_1 a_2 + b_1 c_2) z + (a_1 b_2 + b_1 d_2)}{(c_1 a_2 + d_1 c_2) z + (c_1 b_2 + d_1 d_2)}$$

$$= T_{\begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}}(z).$$

$$\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}$$

$a, b, c, d \in \mathbb{R}$

$\left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \det \gamma = 1 \right\}$

$\| $

$SL_2(\mathbb{R})$

$$\ni \gamma \xrightarrow{T} T_\gamma \in \mathrm{Isom}^+(\mathbb{H}).$$

$$\begin{cases} \underline{T_{\gamma_1 \cdot \gamma_2} = T_{\gamma_1} \circ T_{\gamma_2}} \\ \underline{T_{I_2} = T_{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}(z) = \frac{1 \cdot z + 0}{0 \cdot z + 1} = z = Id(z).} \\ \underline{T_{\gamma^{-1}}(z) = (T_\gamma)^{-1}(z).} \end{cases}$$

(homomorphism of groups)

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad -\gamma = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \qquad \det(-\gamma) = \det(\gamma) = 1.$$

$$T_{-\gamma} = \frac{(-a) \cdot z + (-b)}{(-c) \cdot z + (-d)} = \frac{az + b}{cz + d} = T_\gamma$$

$SL_2(\mathbb{R})$: $\cdot\ \gamma$, $\cdot\ I_2$, $\cdot\ -\gamma$

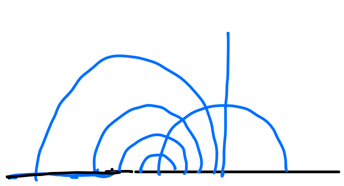
$$SL_2(\mathbb{R}) \xrightarrow{T} Isom^+(\mathbb{H})$$

$$T_\gamma(z) = \frac{az+b}{cz+d}$$

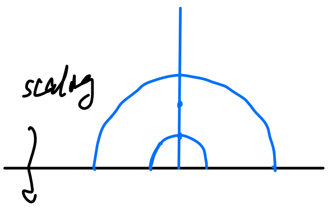
$Isom^+(\mathbb{H})$: $\cdot\ Id$, $\cdot\ T_\gamma = T_{-\gamma}$

$\downarrow$ $\perp$

$Isom^-(\mathbb{H})$



scaling



$Isom^+(\mathbb{H})$
- translation
- rotation
- limit rotation.