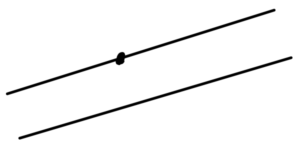
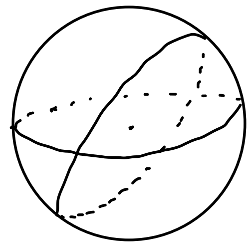
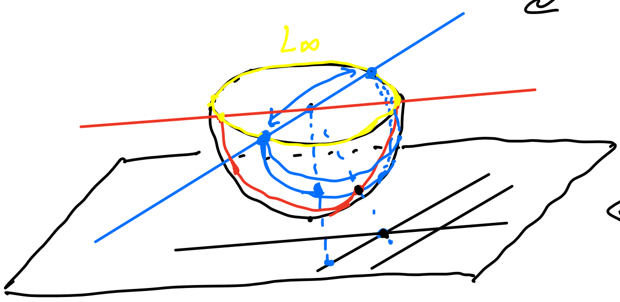
Euclidean Geometry



spherical Geometry

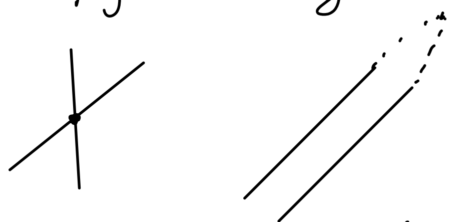


$\mathbb{P}^2 = S^2/\sim$



$\leadsto$

Projective Geometry $\mathbb{P}^2$

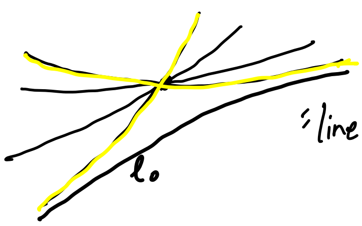


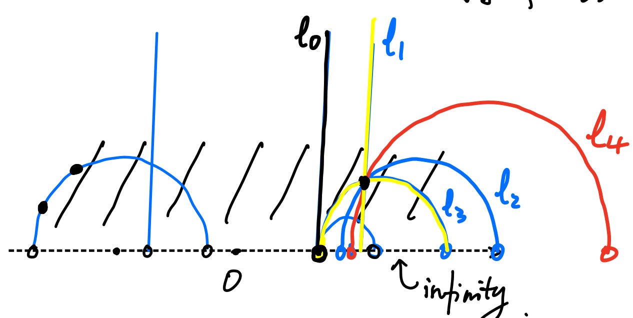
$i = \sqrt{-1}$

hyperbolic geometry : upper half plane $\{z = x+iy,\ \mathrm{Im}(z) > 0\}$

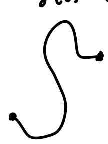
$\parallel$

$\mathbb{H} = \{(x,y) : y > 0\}$





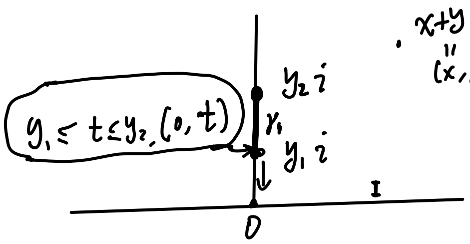
$\gamma(t) = (x(t), y(t))$
$a \le t \le b$



$$L_0(\gamma) = \int_a^b \underbrace{\sqrt{x'(t)^2 + y'(t)^2}\,dt}_{\parallel \atop ds}$$

$x + yi$
$\parallel$
(x, y)

$$\boxed{L(\gamma) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}\,dt}{y(t)}}$$ hyperbolic length.



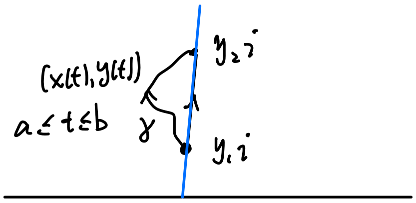
$$L(\gamma_1) = \int_{y_1}^{y_2} \frac{\sqrt{0^2+1^2}}{t}\,dt = \int_{y_1}^{y_2} \frac{1}{t}\,dt = \ln t \Big|_{y_1}^{y_2}$$

$$= \ln y_2 - \ln y_1$$

$$= \ln \frac{y_2}{y_1}$$

$x(t) = 0,\quad y(t) = t$

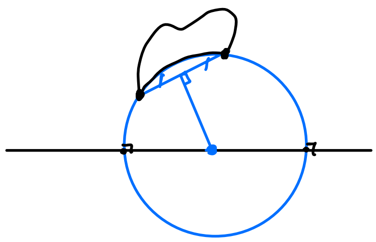
$x'(t) = 0,\quad y'(t) = 1.$



$$\geq \sqrt{y'(t)^2}$$

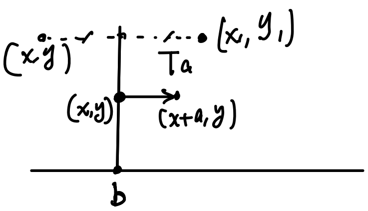
$$\underline{L(\gamma)}=\int_a^b \frac{\sqrt{x'(t)^2+y'(t)^2}}{y(t)}\,dt \geq \int_a^b \frac{\sqrt{y'(t)^2}}{y(t)}\,dt$$

$$\geq \int_a^b \frac{y'(t)\,dt}{y(t)} = \int_{y(a)=y_1}^{y(b)=y_2} \frac{dy}{y} \geq \ln\frac{y_2}{y_1} = \underline{L(\gamma_1)}.$$



$ds_0^2 = dx^2+dy^2$ Euclidean metric.

$ds^2 = \dfrac{dx^2+dy^2}{y^2}$ hyperbolic metric



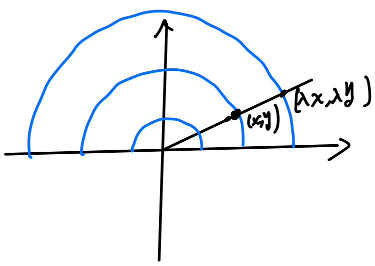
$\underline{T_a}: (x,y) \mapsto (x+a, y) = (u,v)$ $u=x+a$, $du=dx+0=dx$; $v=y$ $dv=dy$

$$\frac{du^2+dv^2}{v^2} = \frac{dx^2+dy^2}{y^2} \Rightarrow T_a \text{ is an isometry}$$

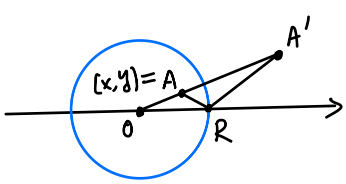
$R_b: (x,y) \mapsto (\overset{u}{2b-x}, \overset{v}{y})$ $\quad \dfrac{du^2+dv^2}{v^2} = \dfrac{(-dx)^2+dy^2}{y^2} = \dfrac{dx^2+dy^2}{y^2}$

$\frac{x_1+x}{2}=b$

$S_\lambda: (x,y) \mapsto (\underset{u}{\lambda x}, \underset{v}{\lambda y})$ $\quad \dfrac{du^2+dv^2}{v^2} = \dfrac{\lambda^2dx^2+\lambda^2dy^2}{\lambda^2y^2} = \dfrac{dx^2+dy^2}{y^2}$



$$A' = |OA'|\cdot\frac{OA'}{|OA'|} = \frac{R^2}{\sqrt{x^2+y^2}}\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right) = R^2\cdot\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right), \quad \frac{OA'}{|OA'|}=\frac{OA}{|OA|}$$



$$|OA|\cdot|OA'| = R^2 \Rightarrow |OA'| = \frac{R^2}{|OA|} = \frac{R^2}{\sqrt{x^2+y^2}}$$

$$\frac{|OA|}{|OR|} = \frac{|OR|}{|OA'|} \Rightarrow \triangle OAR \sim \triangle ORA'$$

$u = R^2\cdot\frac{x}{x^2+y^2}$

$$\Phi(x,y) = R^2\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right) = (u,v).$$

$$du = R^2 \cdot \left(\frac{dx}{x^2+y^2} - \frac{x\cdot(2x\,dx+2y\,dy)}{(x^2+y^2)^2}\right) = R^2 \cdot \frac{1}{(x^2+y^2)^2}\left((x^2+y^2)dx - (2x^2dx + 2xy\,dy)\right)$$

$$= R^2 \cdot \frac{1}{(x^2+y^2)^2}\cdot\left(\underline{(-x^2+y^2)dx - 2xy\,dy}\right)$$

$$dv = R^2 \cdot \frac{1}{(x^2+y^2)^2}\left(\underline{-2xy\,dx + (x^2-y^2)\,dy}\right).$$

$$du^2+dv^2 = R^4 \cdot \frac{1}{(x^2+y^2)^4}\Big(\underline{(x^2-y^2)^2dx^2} - \cancel{2\cdot(-x^2+y^2)dx\cdot 2xy\cdot dy} + \underline{4x^2y^2dy^2}$$
$$\underbrace{+4x^2y^2dx^2}_{\underset{(x^2+y^2)^2dx^2}{\shortparallel}} - \cancel{2\cdot(2xy)\cdot(x^2-y^2)\,dx\,dy} + \underbrace{(x^2-y^2)^2dy^2}_{\underset{(x^2+y^2)^2dy^2}{\shortparallel}}\Big)$$

$$= R^4 \cdot \frac{1}{(x^2+y^2)^2}(dx^2+dy^2).$$

$$\frac{du^2+dv^2}{v^2} = \frac{\cancel{R^4\cdot\frac{1}{(x^2+y^2)^2}}(dx^2+dy^2)}{\cancel{R^4\cdot\frac{1}{(x^2+y^2)^2}}\,y^2} = \frac{dx^2+dy^2}{y^2}.$$