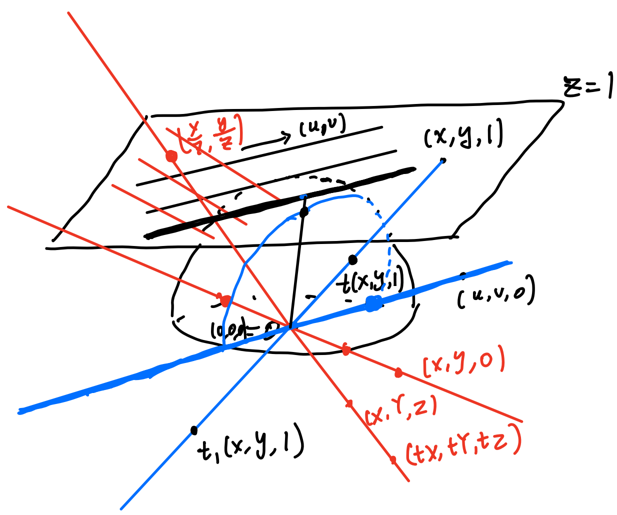
Projective Geometry

$\mathbb{R}^2 \cup L_\infty = \mathbb{P}^2$

$[x, y, z] = [tx, ty, tz]$, $t \neq 0$



$\mathbb{R}^2 \longrightarrow \mathbb{P}^2$

$\underline{(x,y)} \longmapsto [x,y,1] = [tx, ty, t]$, $t \neq 0$

$L_\infty \longrightarrow \mathbb{P}^2$

direction represented by the vector $(u,v) \neq (0,0)$ $\longmapsto [u,v,0] = [tu, tv, 0]$, $t \neq 0$.

homogeneous coordinate

$P = [X, Y, Z] \in \mathbb{P}^2$, $= [tX, tY, tZ]$, $t \neq 0$

- $Z \neq 0$, $\underline{[X,Y,Z] = [\frac{X}{Z}, \frac{Y}{Z}, 1]} \to (\frac{X}{Z}, \frac{Y}{Z}) \in \mathbb{R}^2$, where $x = \frac{X}{Z}$, $y = \frac{Y}{Z}$
- $Z = 0$, $[X, Y, 0] \rightsquigarrow$ direction represented by the vector (X, Y).

line $\subset \mathbb{R}^2$

$l: \ a \cdot x + by + c = 0 \rightsquigarrow a \cdot \frac{X}{Z} + b \cdot \frac{Y}{Z} + c = 0.$

$x = \frac{X}{Z}$, $y = \frac{Y}{Z}$

normal vector (a,b)

$[a, b, c] = [ta, tb, tc]$, $t \neq 0$ $\leftrightsquigarrow$

$\Updownarrow$

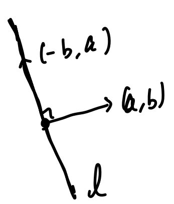
$\boxed{a \cdot X + b \cdot Y + c \cdot Z = 0.}$ equation for a line in $\mathbb{P}^2$.

$a \cdot (tX) + b(tY) + c \cdot (tZ) = 0$

- $Z \neq 0$, $a \cdot \frac{X}{Z} + b \cdot \frac{Y}{Z} + c = 0 \Leftrightarrow a \cdot x + b \cdot y + c = 0$
- $Z = 0$, $a \cdot X + b \cdot Y = 0 \Leftrightarrow \underline{(X, Y) /\!/ (-b, a).}$

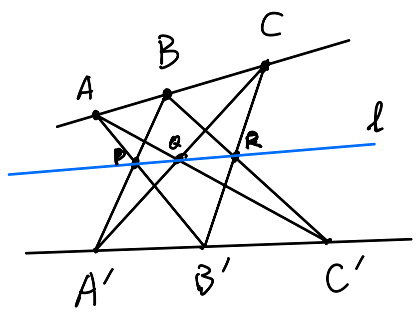
$\Updownarrow$

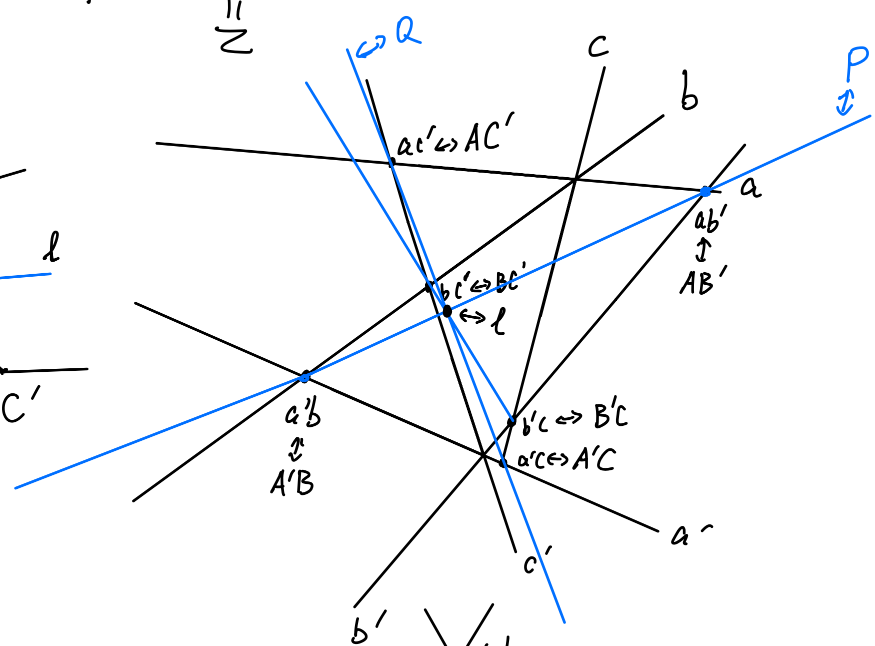
$\frac{X}{Y} = -\frac{b}{a}$

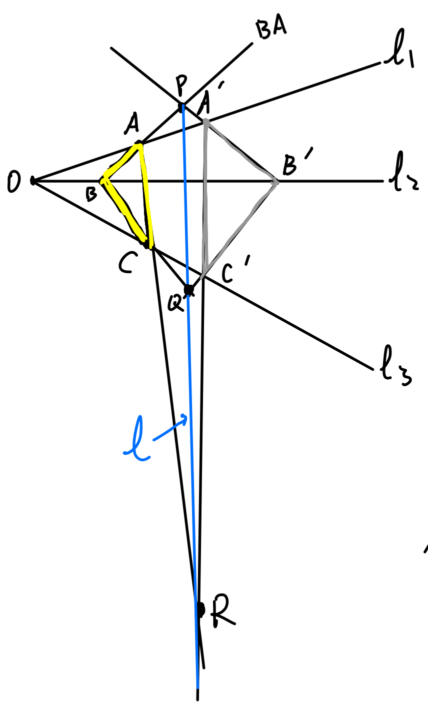


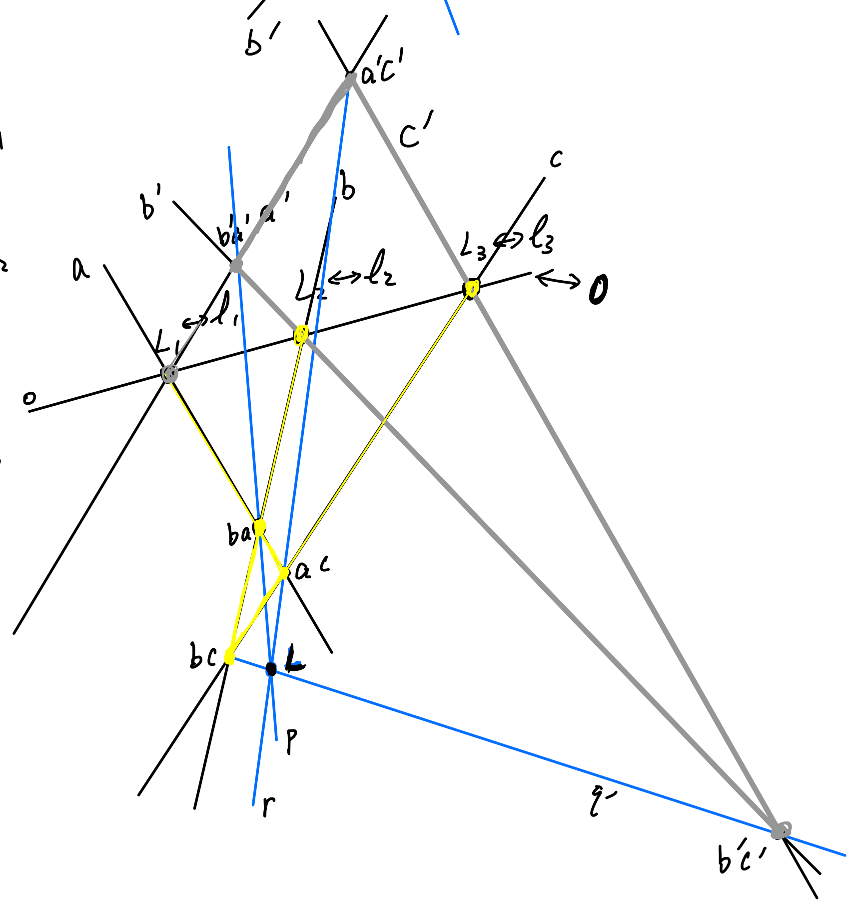
$$P=[X,Y,Z] \in l=[a,b,c] \iff a\cdot X+bY+cZ=0$$
$$\Vert$$
$$[a,b,c]\cdot[X,Y,Z]$$

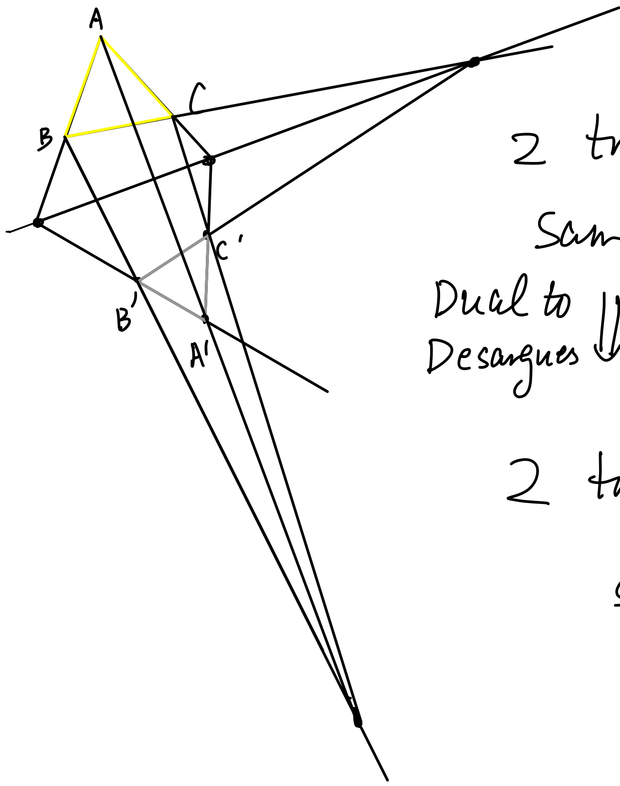
$$l_\infty=[0,0,1]$$
$$\Updownarrow$$
$$\{0\cdot X+0\cdot Y+1\cdot Z=0\}$$
$$\Vert$$
$$Z$$







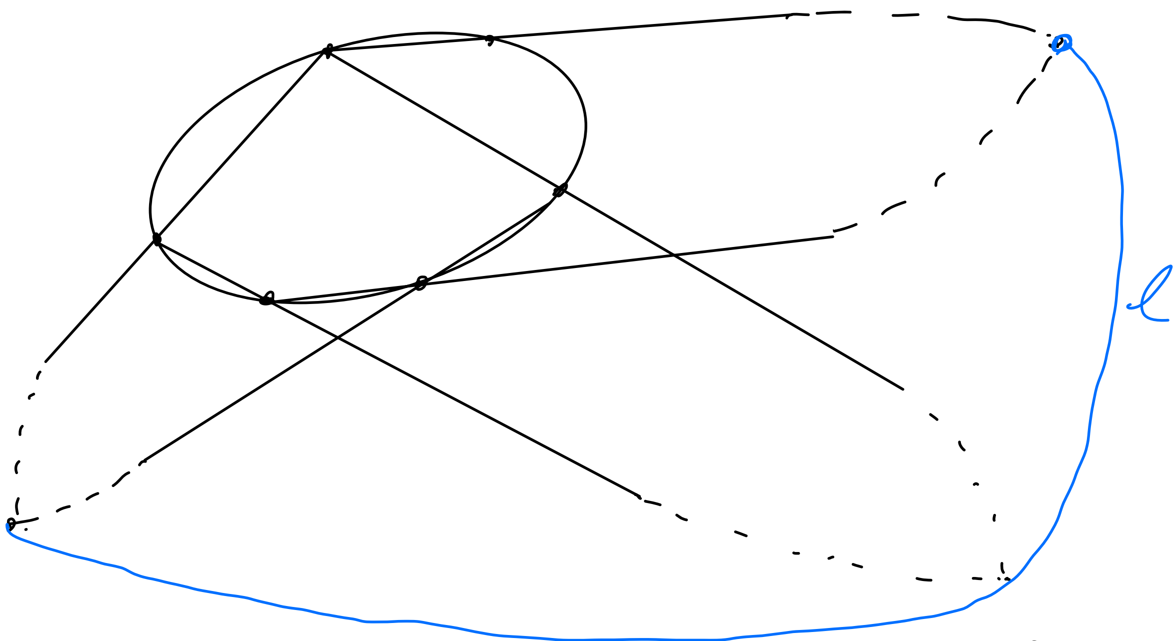




2 triangles in perspective from the same line

Dual to Desargues ⇓ ↕ Dual ⇑ Desargue

2 triangles in perspective from the same point



Pascal

Brianchon