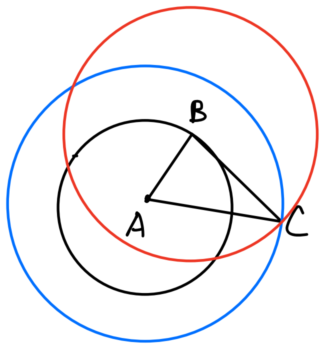
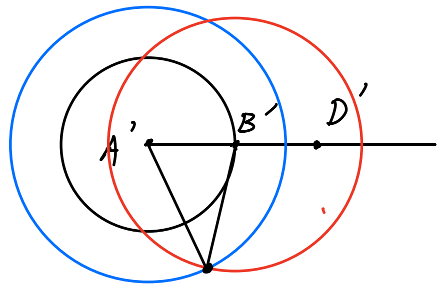
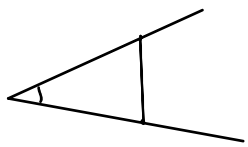
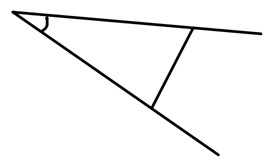
3.15 Lem 3.10

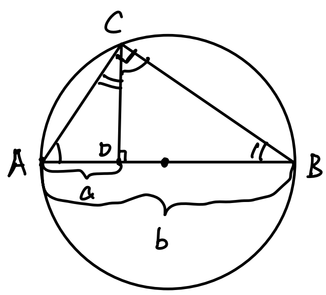








3.26

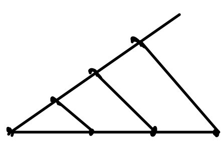


$\triangle ACD \cong \triangle ABC$

$\Rightarrow \dfrac{|AC|}{|AB|} = \dfrac{|AD|}{|AC|}$

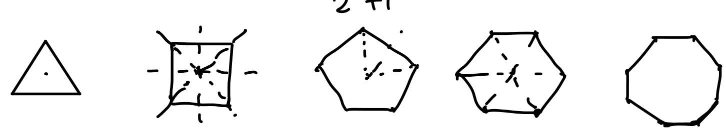
$\Rightarrow |AC|^2 = |AB| \cdot |AD| = b \cdot a$

$\Rightarrow |AC| = \sqrt{a \cdot b}$



$2^{2^1}+1$

3.43.

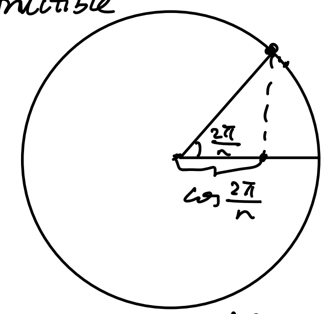


10-gon
‖
2×5 ✓

$n = 2^r \cdot p_1 \cdots p_m$ $\quad p_i = 2^{2^{k_i}}+1$ Fermat prime regular 9-gon is not constructible $\iff$ $\cos\frac{2\pi}{9}$ is not constructible.

$\iff$ $\sin\frac{2\pi}{9}$ is constructible

$$\frac{1}{e^{i\theta}} = e^{-i\theta}$$



$$\omega = e^{i\frac{2\pi}{9}} = \cos\frac{2\pi}{9} + i\cdot\sin\frac{2\pi}{9}.$$

$$\frac{1}{\omega} = \frac{1}{e^{i\frac{2\pi}{9}}} = e^{-i\frac{2\pi}{9}} = \cos\frac{2\pi}{9} - i\cdot\sin\frac{2\pi}{9}.$$

$$x = \omega+\omega^{-1} = 2\cdot\cos\frac{2\pi}{9} = \omega + \frac{1}{\omega}.$$

$$\omega^9 = \left(e^{i\cdot\frac{2\pi}{9}}\right)^9 = e^{i\cdot\frac{2\pi}{9}\cdot 9} = e^{i\cdot 2\pi} = 1.$$

$$0 = \omega^9 - 1 = \underbrace{(\omega-1)\cdot(\omega^8+\omega^7+\omega^6+\omega^5+\omega^4+\omega^3+\omega^2+1)}_{=0}.$$

$$\Rightarrow \quad \omega^4 + \boxed{\omega^3} + \underbrace{\omega^2 + \underbrace{\omega + 1 + \omega^{-1}}_{x} + \omega^{-2}}_{=x^2-2} + \boxed{\omega^{-3}} + \omega^{-4} = 0.$$

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

$$x^2 = \omega^2 + 2 + \omega^{-2}$$

$$x^3 = \omega^3 + 3\cdot\omega^2\cdot\omega^{-1} + 3\omega\cdot\omega^{-2} + \omega^{-3} = \omega^3 + (3\omega+3\omega^{-1}) + \omega^{-3}$$

$$\omega^3+\omega^{-3} = x^3 - 3x.$$

$$x^4 = \omega^4 + 4\cdot\omega^3\cdot\omega^{-1} + 6\cdot\omega^2\omega^{-2} + 4\omega\cdot\omega^{-3} + \omega^{-4}$$

$$= \omega^4 + 4\omega^2 + 6 + 4\omega^{-2} + \omega^{-4}$$

$$= (\omega^4+\omega^{-4}) + \underbrace{4\cdot(x^2-2)+6}_{} = 4x^2 - 2$$

$$a^n - b^n = (a-b)\cdot(a^{n-1} + a^{n-2}\cdot b + \cdots + \cdots + ab^{n-1} + b^n)$$

$$\underbrace{x^4 - 4x^2 + 2}_{\omega^4+\omega^{-4}} + (x^3 - 3x) + (x^2-2) + x + 1 = 0.$$

$$\boxed{x^4 + x^3 - 3x^2 - 2x + 1 = 0} \quad \underset{=\!=}{x=-1} \quad 1-1-3+2+1=0$$

$$x^4 + x^3 - 3x^2 - 3x + x + 1 = \boxed{(x+1)\cdot(x^3-3x+1) = 0} \iff \boxed{x^3 - 3x + 1 = 0}$$

Pentagon: $x = 2\cos\frac{2\pi}{5} \Rightarrow \boxed{x^2+x-1=0} \Rightarrow x = \frac{-1+\sqrt{5}}{2}$

Thm: $x>0$ is a root of an irreducible polynomial of degree n and suppose n has an odd prime factor. Then x is not constructible.

3.52: $\cos\left(\frac{2\pi}{15}\right)$ $\quad 15 = 3\times 5$

3.57: Thm: m, n relatively prime. regular m-gon constructible, n-gon

$\Rightarrow$ $(m\cdot n)$-gon is constructible.

Pf: $\cos\left(\frac{2\pi}{mn}\right)$ $\quad \gcd(m,n) = 1 = m\cdot a + n\cdot b, \ a, b \in \mathbb{Z}$.

$$\begin{aligned} &\cos\left(\frac{2\pi}{mn}\right) \\ &\quad \| \\ &\cos\left(\frac{2\pi}{m\cdot n}\cdot(ma+n\cdot b)\right) = \cos\left(\frac{2\pi}{n}\cdot a + \frac{2\pi}{m}\cdot b\right). \\ &= \cos\left(\frac{2\pi}{n}\cdot a\right)\cdot\cos\left(\frac{2\pi}{m}\cdot b\right) - \sin\left(\frac{2\pi}{n}\cdot a\right)\cdot\sin\left(\frac{2\pi}{m}\cdot b\right) \end{aligned}$$

a polynomial of $\cos\left(\frac{2\pi}{n}\right)$ and $\sin\left(\frac{2\pi}{n}\right)$.

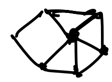
polynomial of $\sin\left(\frac{2\pi}{n}\right)$ and $\cos\left(\frac{2\pi}{n}\right)$ with rational coefficients

$$\boxed{\cos(\alpha+\beta) = \cos\alpha\cdot\cos\beta - \sin\alpha\cdot\sin\beta}$$

5.18 snub cube vertex configuration $(3,3,3,3,4)$

$F_3 = \#$ triangle faces.

$F_4 = \#$ squares.



$$V = 4 \cdot F_4 = \frac{3 \cdot F_3}{4} \Leftrightarrow F_3 = \frac{16 F_4}{3}$$

$$E = \frac{4F_4 + 3F_3}{2} = \frac{4F_4 + 16F_4}{2} = 10 \cdot F_4$$

$$F = F_3 + F_4 = \frac{16F_4}{3} + F_4 = \frac{19}{3} F_4$$

$$2 = \chi = V - E + F = 4F_4 - 10F_4 + \frac{19}{3} F_4 = \frac{1}{3} F_4 \Rightarrow F_4 = 6$$

$$F_3 = \frac{16 \cdot 6}{3} = 32.$$

$5x^2+4xy+2y^2-24x-12y+18=0.$

$\begin{pmatrix}5 & 2\\ 2 & 2\end{pmatrix}=A \qquad |\lambda I-A| = \begin{vmatrix}\lambda-5 & -2\\ -2 & \lambda-2\end{vmatrix} = \underset{\substack{\parallel \\ (\lambda-1)(\lambda-6)}}{\lambda^2-7\lambda+6} = 0$

$\Rightarrow \lambda = 1, 6$

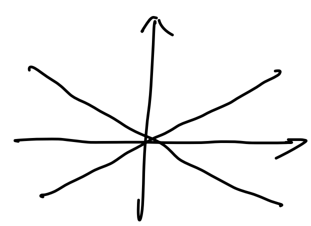
$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1, -1, 0.$

$8x^2+2y^2+8xy-6x-8y+1=0$

$\left|\lambda I-\begin{pmatrix}8 & 4\\ 4 & 2\end{pmatrix}\right| = \begin{vmatrix}\lambda-8 & -4\\ -4 & \lambda-2\end{vmatrix} = \lambda^2-10\lambda+16-16 = \underset{\substack{\parallel \\ \lambda(\lambda-10)}}{\lambda^2-10\lambda}$

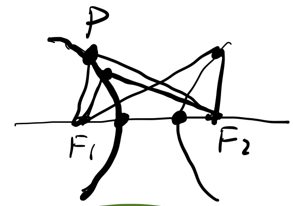
$\Rightarrow \lambda=0, \ \lambda=10. \Rightarrow$ parabola.

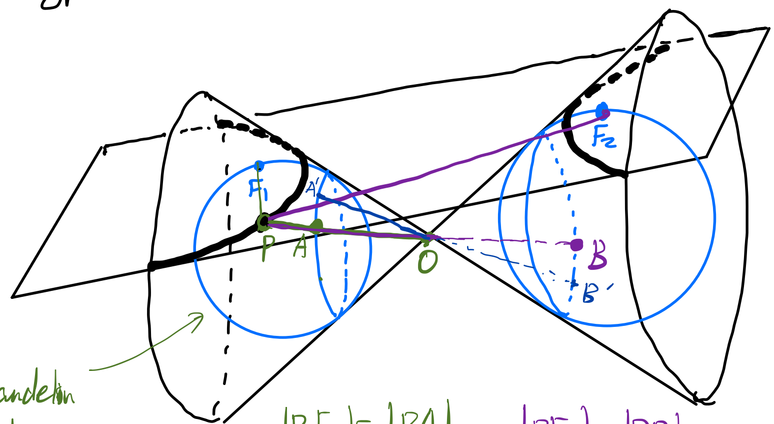
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$



1. Use the following picture to explain why the thickened conic curve is a hyperbola with foci F_1 and F_2.

$|PF_2|-|PF_1| = \text{const}$

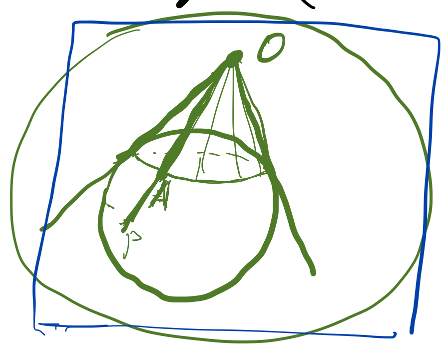




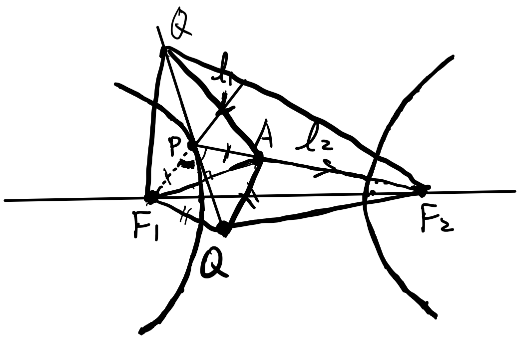
Dandelin sphere

$|PF_1|=|PA|$, $|PF_2|=|PB|$

$|PF_2|-|PF_1| = |PB|-|PA| = |AB|$



2. Give a description of tangents to hyperbola. Then use it to show that if a ray ℓ_1 passes through F_1 (after extension), then its reflection ℓ_2 passes through F_2.



$|PF_2|-|PF_1|$

$\|$

$|PF_2|-|PA| = |AF_2|$

$|QF_2|-|QF_1| < |AF_2|$

$\|$

$|QF_2|-|AQ|$

$\Rightarrow |QF_2|-|QF_1| < |AF_2| = \text{const}.$

$\Rightarrow Q \in$ region between the two branches except for $Q=P$