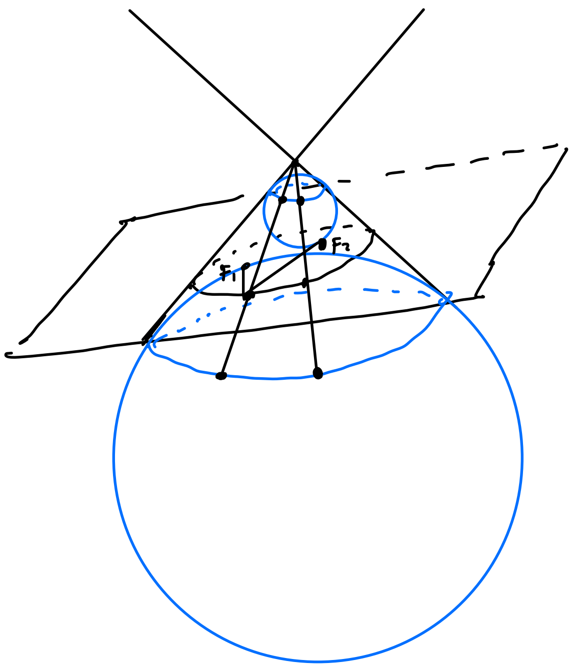
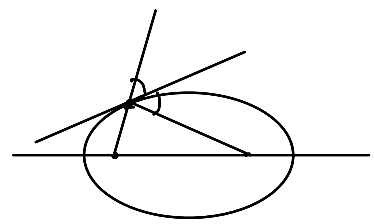
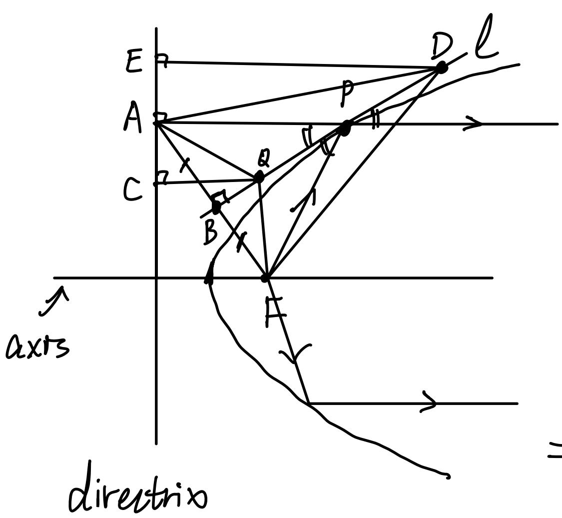
Conic curves

ellipse. parabola, hyperbola.







$$\triangle ABP \overset{SAS}{\cong} \triangle FBP$$

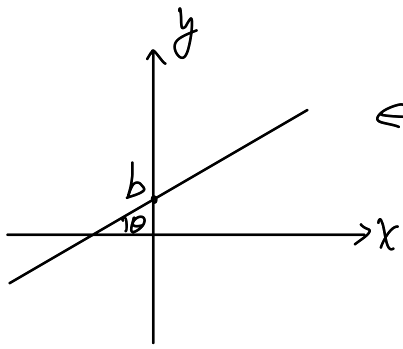
$$\Rightarrow \angle ABQ = \angle FBQ = \frac{\pi}{2}$$

$$|AB| = |FB|.$$

$$\overset{SAS}{\Longrightarrow} \triangle ABQ \cong FBQ$$

$$\Rightarrow \underline{|QF| = |QA|} > |QC|$$

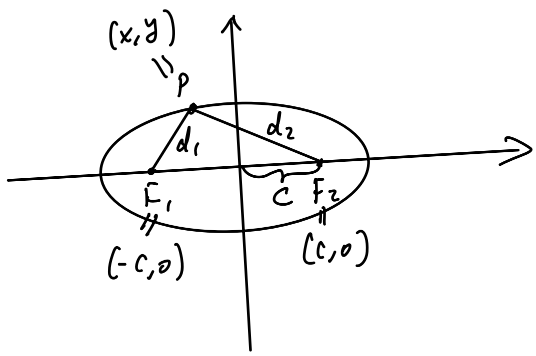
$\Rightarrow$ bisection line l = tangent line of parabola.



$\Leftarrow \quad a\cdot x + b\cdot y = c.$

$y = k\cdot x + b. \quad k = \tan\theta$

Quadratic equation $\rightsquigarrow$ conic curve.



$$d_1 = \sqrt{(x+c)^2 + y^2}$$

$$d_2 = \sqrt{(x-c)^2 + y^2}$$

$$\boxed{d_1 + d_2 = 2a}$$

$$\Downarrow$$

$$\sqrt{(x+c)^2 + y^2} = 2a - \sqrt{(x-c)^2 + y^2}$$

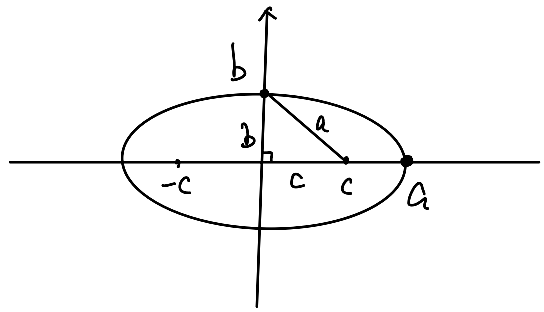
$$\cancel{x^2} + \underline{2c\cdot x} + \cancel{c^2} + \cancel{y^2} = 4a^2 - 4a\cdot\sqrt{(x-c)^2 + y^2} + (\cancel{x^2} \underline{- 2cx} + \cancel{c^2} + \cancel{y^2})$$

$$\Rightarrow \quad \cancel{4}a\sqrt{(x-c)^2 + y^2} = \cancel{4}a^2 - \cancel{4}c\cdot x$$

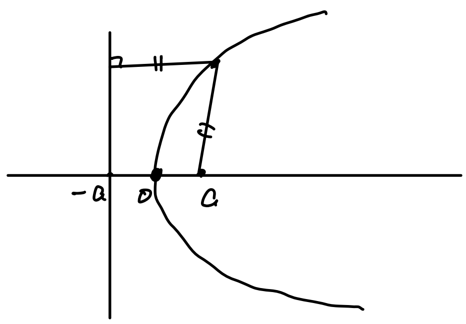
$$\Rightarrow \quad a^2\cdot(x^2 \cancel{- 2cx} + c^2 + y^2) = a^4 - \cancel{2a^2c\cdot x} + c^2x^2.$$

$$\Rightarrow \quad \underbrace{(a^2 - c^2)}_{= b^2}x^2 + a^2\cdot y^2 = a^4 - a^2c^2 = a^2\cdot\underbrace{(a^2 - c^2)}_{= b^2}.$$

$$\Rightarrow \quad b^2x^2 + a^2y^2 = a^2b^2 \Longrightarrow \boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1}$$



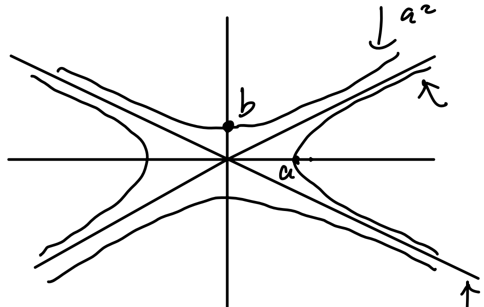
$$(x\ y)\cdot\begin{pmatrix}\frac{1}{a} & 0\\ 0 & \frac{1}{b}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = 1.$$



$$y^2 = 4ax \qquad (x\ y)\cdot\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$$\boxed{(x\ y)\begin{pmatrix}\frac{1}{a} & 0\\ 0 & -\frac{1}{b}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = -1}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \quad (\Rightarrow |y| \geq b)$$



$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (\Rightarrow |x| \geq a$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 = \left(\frac{x}{a} + \frac{y}{b}\right)\cdot\left(\frac{x}{a} - \frac{y}{b}\right)$$

$$\boxed{(Ax^2 + 2Bxy + Cy^2) + 2Dx + 2Ey + F = 0.}$$

$$\shortparallel$$

$$\boxed{(x\ y)\begin{pmatrix}A & B\\ B & C\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}} \qquad A x^3 + Bx^2y + Cxy^2 + Dy^3$$

$$\shortparallel$$

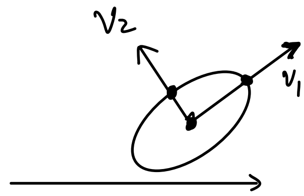
$$M$$

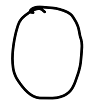
$\underline{Ex}$: $36x^2 - 24xy + 29y^2 + 120x - 290y + 545 = 0$.

$$M = \begin{pmatrix} 36 & -12 \\ -12 & 29 \end{pmatrix}$$

$$\underset{\det(\lambda I - M)}{\underset{\shortparallel}{|\lambda I - M|}} = \underbrace{\begin{vmatrix} \lambda-36 & 12 \\ 12 & \lambda-29 \end{vmatrix}} = \lambda^2 - 65\lambda + \underset{45\times 20}{\underset{\shortparallel}{900}} = (\lambda-20)(\lambda-45) = 0$$

$\Rightarrow$ $\lambda = 20, 45$.





$$\boxed{x^2 + \frac{y^2}{2} = 1}$$

$\bullet$ $\quad x^2 + \frac{y^2}{2} = 0$.

$\emptyset$ $\quad x^2 + \frac{y^2}{2} = -1$.

$\lambda = 20$: $\quad \lambda I - M = \begin{pmatrix} -16 & 12 \\ 12 & -9 \end{pmatrix} \to \begin{pmatrix} 4 & -3 \\ 4 & -3 \end{pmatrix} \to \begin{pmatrix} 4 & -3 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow \underline{w_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix}} \Rightarrow v_1 = \frac{w_1}{\|w_1\|} = \underline{\frac{1}{5}\begin{pmatrix} 3 \\ 4 \end{pmatrix}}$.

$\lambda = 45$: $\lambda I - M = \begin{pmatrix} 9 & 12 \\ 12 & 16 \end{pmatrix} \to \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} \Rightarrow \underline{w_1 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}}$, $\underline{v_2 = \frac{1}{5}\begin{pmatrix} -4 \\ 3 \end{pmatrix}}$.

$M = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$ symmetric matrix.

$\Rightarrow$ eigenvectors associated to different eigenvalues are orthogonal

$\Rightarrow$ M can be diagonalized by orthogonal matrices.

i.e. $\exists$ an orthogonal matrix $\underline{S}$ ($S^T \cdot S = I_2$) s.t.

$$S^{-1} \cdot M S = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \Rightarrow M = S \cdot \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot S^{-1}$$

Ex: $S = (v_1 \ \ v_2) = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix}$. $\quad S^{-1} \cdot M S = \begin{pmatrix} 20 & 0 \\ 0 & 45 \end{pmatrix}$.

$$\underline{36x^2 - 24xy + 29y^2 + 120x - 290y + 545 = 0}.$$

$$\underline{(x \ y) \cdot M \begin{pmatrix} x \\ y \end{pmatrix}} = \underbrace{(x \ y) \cdot S}_{(u \ v)} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \cdot \underbrace{S^{-1} \begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} u \\ v \end{pmatrix}} = \underbrace{\lambda_1 u^2 + \lambda_2 v^2}_{20u^2 + 45v^2}$$

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = S \cdot \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{3}{5} & -\frac{4}{5} \\ \frac{4}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}} \Leftrightarrow \begin{cases} x = \frac{3}{5}u - \frac{4}{5}v \\ y = \frac{4}{5}u + \frac{3}{5}v \end{cases}.$$

$$120x - 290y = (120 \ \ -290) \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \underline{(120 \ -290) \cdot S} \begin{pmatrix} u \\ v \end{pmatrix} = (-160 \ \ -270) \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\Rightarrow \quad 20u^2+45v^2 \underline{-160u-270v}+545=0.$$

$$\Updownarrow$$

$$4u^2+9v^2-\underline{32u-54v}+109=0$$

$$4(\underbrace{u^2-8u}_{+16}) + 9(\underbrace{v^2-6v}_{+9}) = -109\underbrace{+64+81}_{=145} \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\boxed{4(u-4)^2+9(v-3)^2=36}$$

$$\frac{(u-4)^2}{9_{=3^2}}+\frac{(v-3)^2}{4_{=2^2}}=1.$$