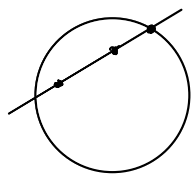
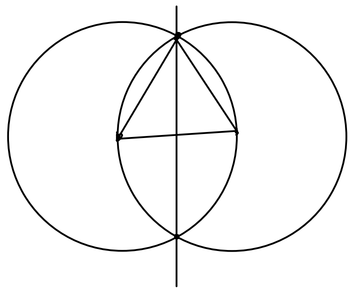
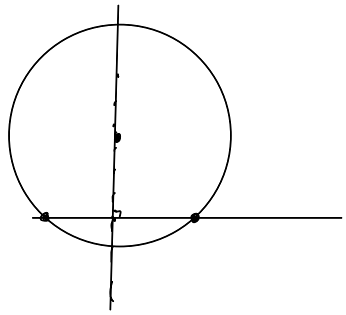
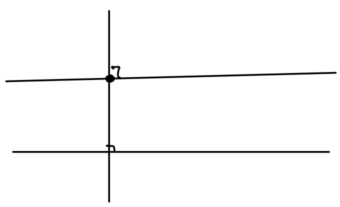
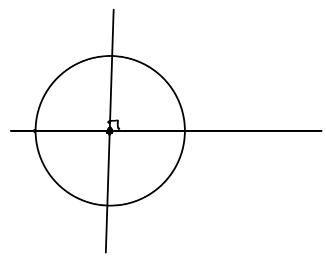
Compass / straightedge Construction.

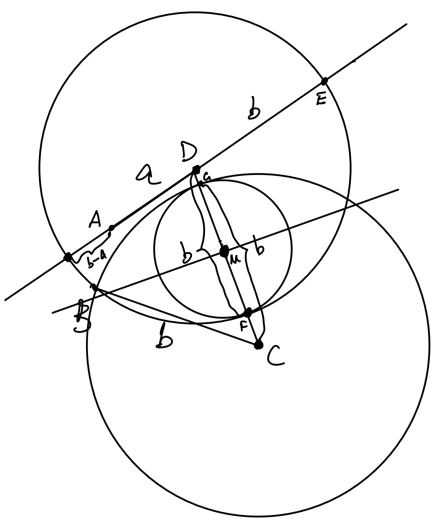


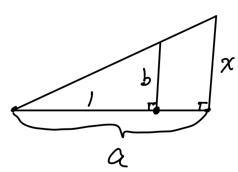




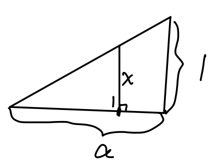








$$\frac{b}{1} = \frac{x}{a} \Rightarrow x = a \cdot b.$$

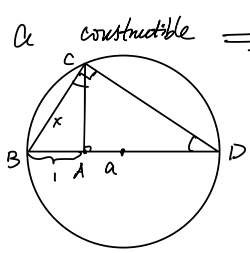


(1) $\mapsto$ $n > 0$ integer

$\mapsto$ $\frac{1}{n}$ $\mapsto$ $\frac{m}{n}$ rational numbers.

$$\frac{x}{1} = \frac{1}{a} \Rightarrow x = \frac{1}{a}.$$

- a constructible $\Rightarrow$ $\sqrt{a}$ is constructible.

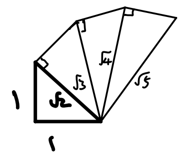


$\triangle ABC \sim \triangle CBD$

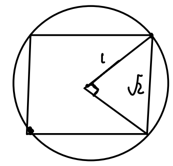
$$\Rightarrow \frac{|AB|}{|CB|} = \frac{|BC|}{|BD|} \Leftrightarrow \frac{1}{x} = \frac{x}{a} \Rightarrow x^2 = a$$

$$x = \sqrt{a}.$$

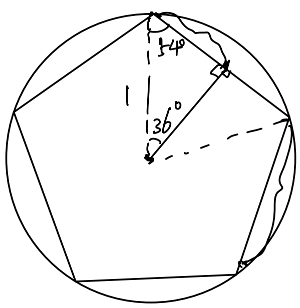
$1 \rightsquigarrow \sqrt{5}$



Square



regular pentagon.



$\frac{a+b\sqrt{5}}{c}$

regular n polygon: sum interior angle

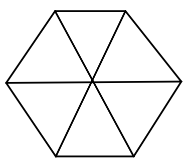
$\parallel$

$(n-2)\cdot 180°$

$\frac{n-2}{n}\cdot 180° = \frac{(n-2)\pi}{n}$

$n=5$. $\frac{3\pi}{5} = \frac{3}{5}\times 180° = 108°$

$\cos 54° = \cos\frac{3\pi}{10}$



Thm: regular n-polygon is constructible $\Leftrightarrow$

$n = 2^r \cdot P_1 \cdot P_2 \cdots P_k$

where $P_i = 2^{2^{k_i}}+1$ is Fermat prime.

$\parallel$

F_{k_i}

$F_0=3$, $F_1=5$, $F_2=17$, $F_3=257$, $F_4=65537$.

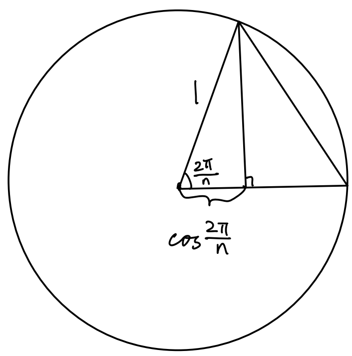
$\parallel$

$2^{2^4}+1$

$F_k = 2^{2^k}+1$. $641 \mid F_5$.

$5 \le k \le 20$?

construct regular n-polygon $\Longleftrightarrow$ construct $\cos\left(\frac{2\pi}{n}\right)$



<u>Thm</u>: If m and n are relatively prime, and we can construct regular m-gon and regular n-gon, then we can construct regular mn-gon.

<u>Pf</u>: m, n relatively prime $\Leftrightarrow \gcd(m,n)=1 \Rightarrow \exists\, a, b \in \mathbb{Z}$ s.t. $am+bn=1$

$$\Rightarrow \quad b\cdot\frac{2\pi}{m} + a\cdot\frac{2\pi}{n} = \frac{2\pi}{mn}\cdot(bn+am) = \frac{2\pi}{mn}.$$

$$\Rightarrow \quad \cos\left(\frac{2\pi}{mn}\right) = \cos\left(b\cdot\frac{2\pi}{m} + a\cdot\frac{2\pi}{n}\right) = \cos\left(b\cdot\frac{2\pi}{m}\right)\cdot\cos\left(a\cdot\frac{2\pi}{n}\right) - \sin\left(b\cdot\frac{2\pi}{m}\right)\sin\left(a\cdot\frac{2\pi}{n}\right)$$

regular m-gon constructible $\Rightarrow \cos\left(\frac{2\pi}{m}\right)$ constructible $\Rightarrow \cos\left(b\cdot\frac{2\pi}{m}\right)$ is constructible

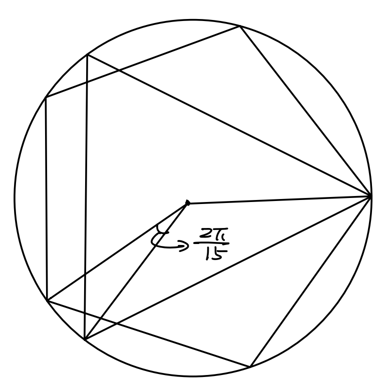
$\sqrt{1-\cos^2\left(b\frac{2\pi}{m}\right)} = \sin\left(b\cdot\frac{2\pi}{m}\right)$ is constructible

Similarly for n-gon.

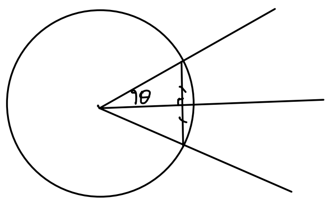
$\Rightarrow \cos\left(\frac{2\pi}{mn}\right)$ is constructible.

<u>Ex</u>: $m=3$, $n=5$. $a=-3$, $n=2$.

$$\frac{2\pi}{15} = 2\cdot\frac{2\pi}{3} - 3\cdot\frac{2\pi}{5}$$



- bisecting an angle:



$\cos(2\theta) = 2\cdot\cos^2\theta - 1 \iff \cos(\theta) = \sqrt{\frac{1+\cos(2\theta)}{2}}$

constructible from $\cos(2\theta)$.

$\Updownarrow$

quadratic equations solvable by compass/straightedg.

Trisecting an angle: $\cos(3\theta) = 4\cos^3\theta - 3\cos\theta$.

solving cubic equation involes taking cubic root $\Rightarrow$ not constructible by compass / straight edge.

- Field of constructible numbers.

Axioms

Field: $(S, +, \cdot)$

- $a+b=b+a$, $a+0=a$, $a+(-a)=0$. associative
- $a\cdot b = b\cdot a$, $a\cdot 1 = a$. $a\cdot\frac{1}{a}=1$ for $a\neq 0$. associative
- distributive: $a\cdot(b+c) = a\cdot b + a\cdot c$.

Constructible numbers form a field. between $\mathbb{Q}$ and $\mathbb{R}$.

Ex:
- $\sqrt{\sqrt{\frac{2}{7}} + 3\cdot\sqrt{5}} - \sqrt[4]{11}$ constructible
- $\cos\left(\frac{2\pi}{7}\right)$ is not constructible.