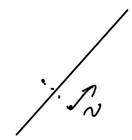
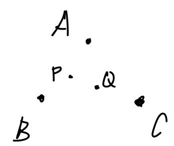
$f: \mathbb{R}^2 \to \mathbb{R}^2$ isometry : $|f(P)\,f(Q)| = |PQ| \quad \forall P, Q$

1. translation  2. rotation  3. reflection  4. glide reflection



Thm A: Let $A, B, C$ be 3 pts. not colinear
$\Updownarrow$
(not on a line)

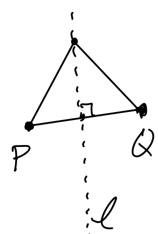



Any point $P \in \mathbb{R}^2$ is uniquely determined by its distances to $A, B, C$.

In other words, $\left.\begin{array}{l} |PA| = |QA| \\ |PB| = |QB| \\ |PC| = |QC| \end{array}\right\} \Rightarrow P = Q.$

( does not hold $\Leftarrow P \neq Q$ )

Fact:

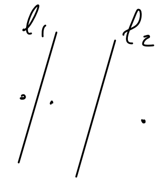



Pf: Proof by contradiction. Suppose $P \neq Q$.

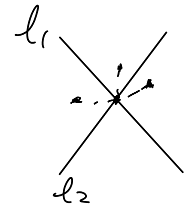
$$\begin{cases} |PA| = |QA| \Rightarrow A \in l \\ |PB| = |QB| \Rightarrow B \in l \\ |PC| = |QC| \Rightarrow C \in l \end{cases}$$

contradiction. So $P = Q$

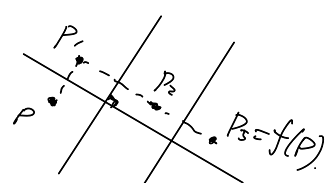
Thm A
$\Downarrow$
Thm (3 reflection thm): Any isometry of $\mathbb{R}^2$ is a composition of at most 3 reflections.




translation



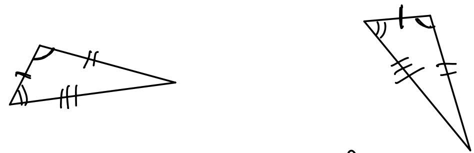

rotation.



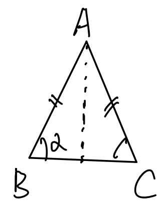

(Stillwell: Four Pillars of Geometry)

- Congruence of triangles.

SSS $\qquad$ SAS $\qquad$ ASA



Thm (Pons Asinorum)

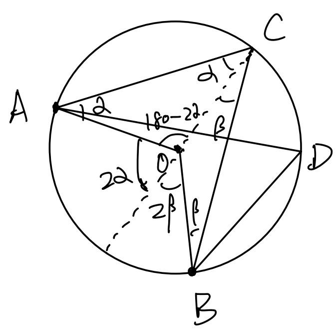



$\triangle ABC \qquad \triangle ACB$

$$\left\{\begin{array}{l} |AB| = |AC| \\ \angle BAC = \angle CAB \\ |AC| = |AB| \end{array}\right.$$

$\Downarrow$ SAS

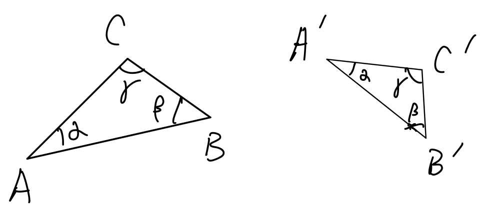
$\angle ABC = \angle ACB \Leftarrow \triangle ABC \cong \triangle ACB$

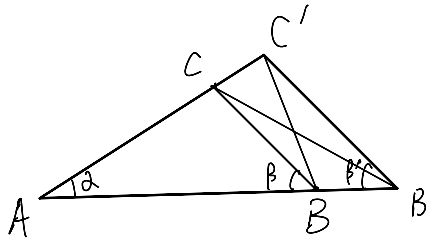



$\angle AOB = 2 \cdot \angle ACB$ (star trek lem).

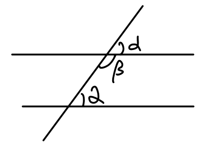
$\angle ACB = \angle ADB = \frac{1}{2} \angle AOB$.

- Similar triangles.






$\Rightarrow \quad BC \parallel B'C'$

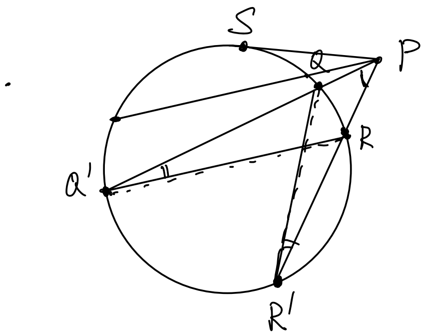



(Thale's Thm).

$$\Rightarrow \quad \frac{|AB|}{|AB'|} = \frac{|AC|}{|AC'|} \quad \Leftarrow \quad |\triangle AB'C| = |\triangle ABC| + \underbrace{|\triangle BCB'|}_{\parallel}$$

$$|\triangle ABC'| = |\triangle ABC| + \underline{|\triangle BCC'|}$$

$$\underset{\dfrac{|\triangle ABC|}{|\triangle AB'C|}}{\parallel} \qquad \underset{\dfrac{|\triangle ABC|}{|\triangle ABC'|}}{\parallel}$$



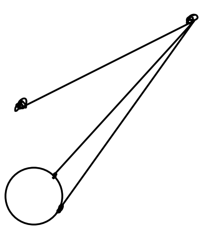

power of P w.r.t. the circle

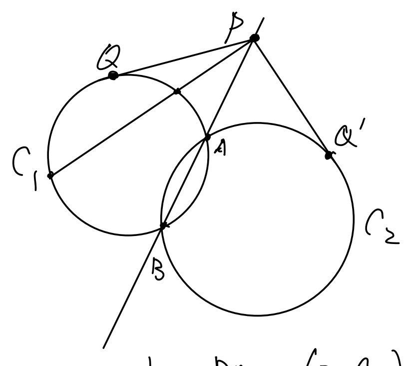
$$\parallel$$

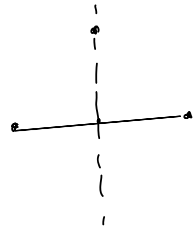
$$\underline{|PQ| \cdot |PQ'| = |PR| \cdot |PR'| = |PS|^2}$$

$$\underline{\triangle PQR'} \sim \underline{\triangle PRQ'}.$$

$$\Rightarrow \quad \frac{|PQ|}{|PR|} = \frac{|PR'|}{|PQ'|} \qquad \checkmark.$$

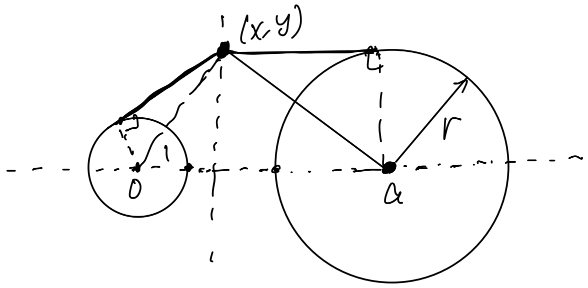


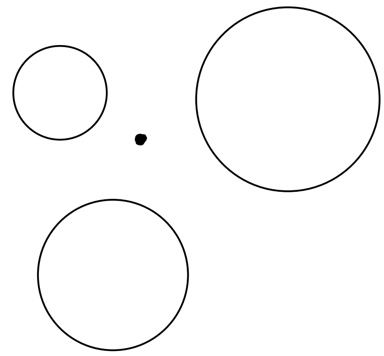


$$\text{power}(P, C_1) = \underbrace{|PA| \cdot |PB|}_{\parallel} = \text{power}(P, C_2).$$

$$|PQ|^2 = |PQ'|^2$$

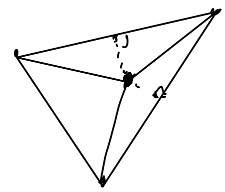


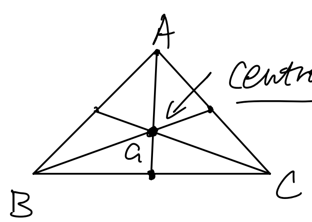
$$x^2+y^2-1^2 = (x-a)^2+y^2-r^2 = x^2-2ax+a^2+y^2-r^2$$

$$-1 = -2ax+a^2-r^2$$

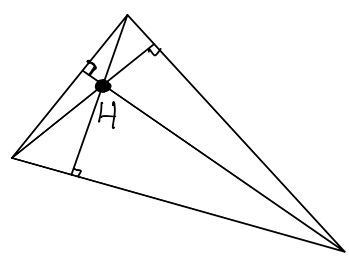
$$\Rightarrow \quad x = \frac{1+a^2-r^2}{2a}$$

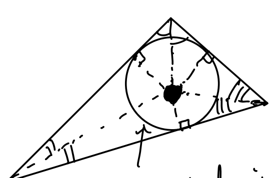


- Medians and Centroid.

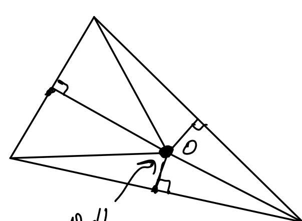



centroid = center of mass



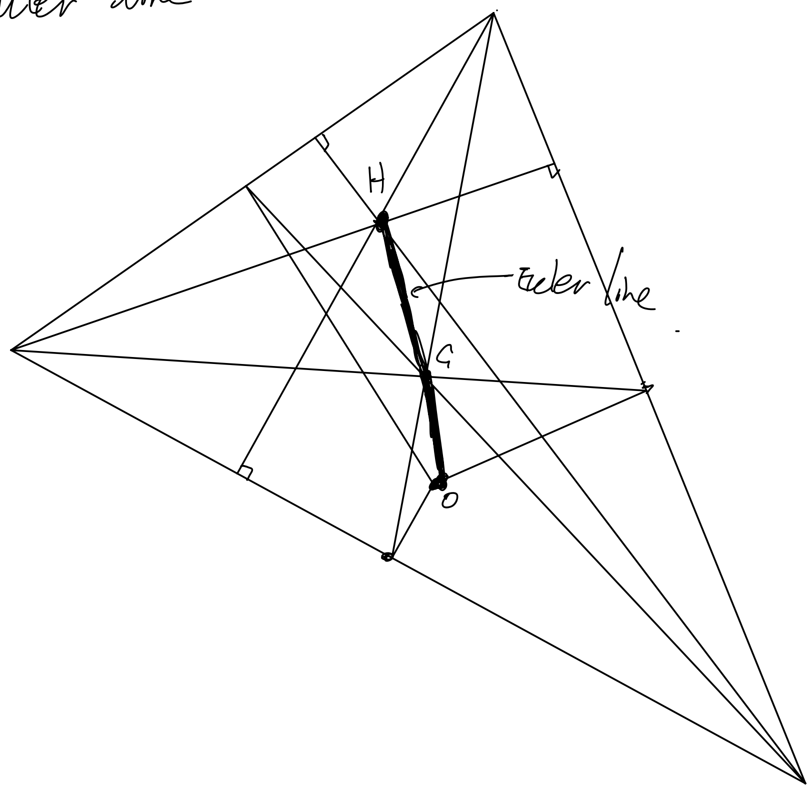



inscribed circle.

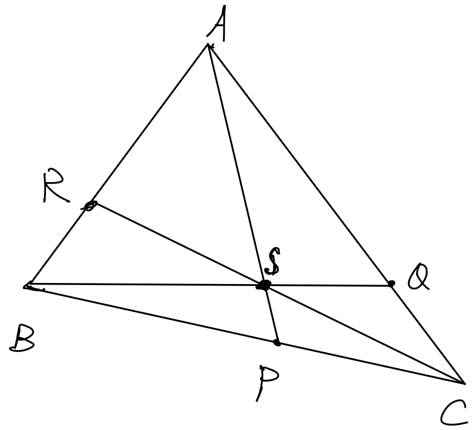



ortho-center.

Euler line




Cova Thm




AP, BQ and CR pass through a common point S

if and only if:

$$\frac{|AR|}{|RB|} \cdot \frac{|BP|}{|PC|} \cdot \frac{|CQ|}{|QA|} = 1.$$