4.1 : 2.

Maximize $10x_1 + 13x_2 + 8x_3$

Subject to

$$x_1 + 1.2x_2 + 0.7x_3 \le 600$$
$$0.5x_1 + 0.5x_2 + 0.3x_3 \le 300$$
$$0.7x_1 + 0.7x_2 + 0.3x_3 \le 300$$
$$0.7x_3 \le 140$$
$$x_i \ge 0,\ i = 1,2,3 \text{ integral.}$$

$\Leftrightarrow$

$$10x_1 + 12x_2 + 7x_3 \le 6000$$
$$5x_1 + 5x_2 + 3x_3 \le 3000$$
$$7x_1 + 7x_2 + 3x_3 \le 3000$$
$$x_3 \le 200$$
$$x_i \ge 0,\ i = 1,2,3 \text{ integral}$$

Using cutting plane method, need 3 cutting planes to get optimal solution:
(see the code at the end)

$(x_1, x_2, x_3) = (0, 343, 199)$

$$\begin{array}{c|cccccccc} & x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \\ \hline x_4 & 10 & 12 & 7 & 1 & 0 & 0 & 0 & 6000 \\ x_5 & 5 & 5 & 3 & 0 & 1 & 0 & 0 & 3000 \\ x_6 & 7 & 7 & 3 & 0 & 0 & 1 & 0 & 3000 \\ x_7 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 200 \\ \hline & -10 & -13 & -8 & 0 & 0 & 0 & 0 & 0 \end{array}$$

$\rightarrow$

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{12}{7} & -\frac{13}{7} & \frac{3400}{7} \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{7} & -\frac{6}{7} & \frac{4800}{7} \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{7} & -\frac{3}{7} & \frac{2400}{7} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 200 \\ 3 & 0 & 0 & 0 & 0 & \frac{13}{7} & \frac{17}{7} & \frac{42400}{7} \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{12}{7} & -\frac{13}{7} & 0 & \frac{3400}{7} \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{7} & -\frac{6}{7} & 0 & \frac{4800}{7} \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{7} & -\frac{3}{7} & 0 & \frac{2400}{7} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 200 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & 1 & \frac{6}{7} \\ 3 & 0 & 0 & 0 & 0 & \frac{13}{7} & \frac{17}{7} & 0 & \frac{42400}{7} \end{pmatrix}$$

← cutting plane 1

$\rightarrow$

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{5}{4} & 0 & -\frac{13}{4} & \frac{977}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{3}{2} & 687 \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{3}{4} & \frac{687}{2} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} & 0 & \frac{7}{4} & \frac{397}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 1 & -\frac{7}{4} & \frac{3}{2} \\ 3 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & \frac{17}{4} & \frac{12107}{2} \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{5}{4} & 0 & -\frac{13}{4} & 0 & \frac{977}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{3}{2} & 0 & 687 \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{3}{4} & 0 & \frac{687}{2} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} & 0 & \frac{7}{4} & 0 & \frac{397}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 1 & -\frac{7}{4} & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{13}{4} & 1 & -\frac{1}{2} \\ 3 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & \frac{17}{4} & 0 & \frac{12107}{2} \end{pmatrix}$$

← cutting plane 2

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 489 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{13} & 0 & 0 & -\frac{6}{13} & \frac{8934}{13} \\ 1 & 1 & 0 & 0 & 0 & \frac{4}{13} & 0 & 0 & -\frac{3}{13} & \frac{4467}{13} \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{13} & 0 & 0 & \frac{7}{13} & \frac{2577}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} & 1 & 0 & -\frac{7}{13} & \frac{23}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{13} & 0 & 1 & -\frac{4}{13} & \frac{2}{13} \\ 3 & 0 & 0 & 0 & 0 & \frac{12}{13} & 0 & 0 & \frac{17}{13} & \frac{78687}{13} \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 489 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{13} & 0 & 0 & -\frac{6}{13} & 0 & \frac{8934}{13} \\ 1 & 1 & 0 & 0 & 0 & \frac{4}{13} & 0 & 0 & -\frac{3}{13} & 0 & \frac{4467}{13} \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{13} & 0 & 0 & \frac{7}{13} & 0 & \frac{2577}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} & 1 & 0 & -\frac{7}{13} & 0 & \frac{23}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{13} & 0 & 1 & -\frac{4}{13} & 0 & \frac{2}{13} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4}{13} & 0 & 0 & -\frac{24}{65} & 1 & -\frac{8}{13} \\ 3 & 0 & 0 & 0 & 0 & \frac{12}{13} & 0 & 0 & \frac{17}{13} & 0 & \frac{78687}{13} \end{pmatrix}$$

← cutting plane 3

$\rightarrow$

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & -\frac{13}{4} & 491 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{5}{4} & 688 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{5} & 1 & 343 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{5}{4} & 199 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{6}{5} & -\frac{13}{4} & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 3 & 6051 \end{pmatrix}$$

$(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$

$=$

$(0, 343, 199, 491, 688, 2, 1)$

Maximum $= 6051$

$=$

$343 \times 13 + 199 \times 8$

5.1: 6

Tableau 5.32

	5	7	9	0	
	(100)	0	0	(20)	120
	6	7	10	0	
basic variable →	(0)	(60)	(70)	0	140
	100	60	80	20	Supply

Demand

$$\left.\begin{array}{l} v_1+w_1=5 \\ v_1+w_4=0 \\ v_2+w_1=6 \\ v_2+w_2=7 \\ v_2+w_3=10 \end{array}\right\} \Rightarrow \begin{array}{l} v_2=0 \\ w_1=6 \\ w_2=7 \\ w_3=10 \\ v_1=5-6=-1 \\ w_4=0-(-1)=1 \end{array} \Rightarrow \begin{array}{l} x_{12}:\ v_1+w_2-c_{12}=-1+7-7=-1 \\ x_{13}:\ v_1+w_3-c_{13}=-1+10-9=0 \\ x_{24}:\ v_2+w_4-c_{24}=0+1-0=1. \end{array}$$

$$\left\{\begin{pmatrix} bp & -1 & 0 & bp \\ bp & bp & bp & (1) \end{pmatrix}, \begin{pmatrix} 100 & 0 & 0 & 20 \\ (0) & 60 & 80 & 0 \end{pmatrix}\right\}$$

(1): entering; (0): leaving

⇝ Loop: $(2,4) \to (2,1) \to (1,1) \to (1,4)$

Pivot ⟹

$$\left\{\begin{pmatrix} bp & 0 & (1) & bp \\ -1 & bp & bp & bp \end{pmatrix}, \begin{pmatrix} 100 & 0 & 0 & (20) \\ 0 & 60 & 80 & 0 \end{pmatrix}\right\}$$

⇝ Loop: $(1,3) \to (1,4) \to (2,4) \to (2,3)$

Pivot ⟹

$$\left\{\begin{pmatrix} bp & -1 & bp & -1 \\ 0 & bp & bp & bp \end{pmatrix}, \begin{pmatrix} 100 & 0 & 20 & 0 \\ 0 & 60 & 60 & 20 \end{pmatrix}\right\}$$

⇑ optimal solution

```
]=
{{{1, 1}, {1, 3}, {2, 4}, {2, 2}, {2, 3}}, {100, 20, 20, 60, 60}, 1700}
```

1700 ↑ Minimal cost.

$100 \cdot 5 + 20 \cdot 9 + 60 \cdot 7 + 6 \cdot 10$

$=$

$500 + 180 + 420 + 600 = 1700.$

5.1: 8

2 (50)	5 0	6 0	3 0	0 (50)	100
9 0	6 0	2 0	1 (90)	0 0	90
7 (20)	7 (50)	2 (30)	4 (30)	0 0	130
70	50	30	120	50	

$\sum s_i = 100 + 90 + 130 = 320$ ✓

$\sum d_j = 70 + 50 + 30 + 120 = 270$

$v_1 + w_1 = 2$
$v_1 + w_5 = 0$
$v_2 + w_4 = 1$
$v_3 + w_1 = 7$
$v_3 + w_2 = 7$
$v_3 + w_3 = 2$
$v_3 + w_4 = 4$

$\Rightarrow$

$v_3 = 0$
$w_1 = 7$
$w_2 = 7$
$w_3 = 2$
$w_4 = 4$
$v_1 = 2 - 7 = -5$
$v_2 = 1 - 4 = -3$
$w_5 = -(-5) = 5$

$\Rightarrow$

$x_{12}: v_1 + w_2 - c_{12} = -5 + 7 - 5 = -3$
$x_{13}: v_1 + w_3 - c_{13} = -5 + 2 - 6 = -9$
$x_{14}: v_1 + w_4 - c_{14} = -5 + 4 - 3 = -4$
$x_{21}: v_2 + w_1 - c_{12} = -3 + 7 - 9 = -5$
$x_{22}: v_2 + w_2 - c_{22} = -3 + 7 - 6 = -2$
$x_{23}: v_2 + w_3 - c_{23} = -3 + 2 - 2 = -3$
$x_2: v_2 + w_5 - c_{25} = -3 + 5 - 0 = 2$
$x_{35}: v_3 + w_5 - c_{35} = 0 + 5 - 0 = 5.$

$$\left\{\begin{pmatrix} bp & -3 & -9 & -4 & bp \\ -5 & -2 & -3 & bp & 2 \\ bp & bp & bp & bp & (5) \end{pmatrix}, \begin{pmatrix} 50 & 0 & 0 & 0 & 50 \\ 0 & 0 & 0 & 90 & 0 \\ (20) & 50 & 30 & 30 & 0 \end{pmatrix}\right\}$$

$\rightsquigarrow$ Loop: $(3,5) \to (3,1) \to (1,1) \to (1,5)$

$$\left\{\begin{pmatrix} bp & (2) & -4 & 1 & bp \\ -10 & -2 & -3 & bp & -3 \\ -5 & bp & bp & bp & bp \end{pmatrix}, \begin{pmatrix} 70 & 0 & 0 & 0 & (30) \\ 0 & 0 & 0 & 90 & 0 \\ 0 & 50 & 30 & 30 & 20 \end{pmatrix}\right\}$$

$\rightsquigarrow$ Loop: $(1,2) \to (1,5) \to (3,5) \to (3,2)$

$$\left\{\begin{pmatrix} bp & bp & -6 & -1 & -2 \\ -8 & -2 & -3 & bp & -3 \\ -3 & bp & bp & bp & bp \end{pmatrix}, \begin{pmatrix} 70 & 30 & 0 & 0 & 0 \\ 0 & 0 & 0 & 90 & 0 \\ 0 & 20 & 30 & 30 & 50 \end{pmatrix}\right\}$$

↑ optimal solution

```
{{{1, 1}, {1, 2}, {2, 4}, {3, 5}, {3, 2}, {3, 3}, {3,
  4}}, {70, 30, 90, 50, 20, 30, 30}, (700)}
```

↓ minimal cost.

$70 \cdot 2 + 30 \cdot 5 + 90 \cdot 1 + 20 \cdot 7 + 30 \cdot 2 + 30 \cdot 4 + 0$

$= 140 + 150 + 90 + 140 + 60 + 120 = 700$

16. Dual of the mxn transportation problem:

Maximize $\sum_{i=1}^{m} s_i \cdot v_i + \sum_{j=1}^{n} d_j \cdot w_j$

Subject to $v_i + w_j \leq c_{ij}$. $1 \leq i \leq m$, $1 \leq n \leq n$

v_i, w_j no restricted.

In[62]:=
```
Pivot[A_, k_, l_] := Module[{B = A},
m = Dimensions[A][[1]]; n = Dimensions[A][[2]];
For[i = 1, i ≤ m, i++,
If[i == k,
B[[i]] = A[[i]] / A[[k, l]],
 B[[i]] = -A[[i, l]] / A[[k, l]] * A[[k]] + A[[i]]]
];
B
]
```

In[40]:=
```
A1 = {{10, 12, 7, 1, 0, 0, 0, 6000}, {5, 5, 3, 0, 1, 0, 0, 3000},
   {7, 7, 3, 0, 0, 1, 0, 3000}, {0, 0, 1, 0, 0, 0, 1, 200}, {-10, -13, -8, 0, 0, 0, 0, 0}};
MatrixForm[A1]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} 10 & 12 & 7 & 1 & 0 & 0 & 0 & 6000 \\ 5 & 5 & 3 & 0 & 1 & 0 & 0 & 3000 \\ 7 & 7 & 3 & 0 & 0 & 1 & 0 & 3000 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 200 \\ -10 & -13 & -8 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[63]:=
```
MatrixForm[A2 = Pivot[A1, 3, 2]]
```

Out[63]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & \frac{13}{7} & 1 & 0 & -\frac{12}{7} & 0 & \frac{6000}{7} \\ 0 & 0 & \frac{6}{7} & 0 & 1 & -\frac{5}{7} & 0 & \frac{6000}{7} \\ 1 & 1 & \frac{3}{7} & 0 & 0 & \frac{1}{7} & 0 & \frac{3000}{7} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 200 \\ 3 & 0 & -\frac{17}{7} & 0 & 0 & \frac{13}{7} & 0 & \frac{39\,000}{7} \end{pmatrix}$$

In[64]:=
```
{MatrixForm[A3 = Pivot[A2, 4, 3]],
MatrixForm[Round[A3, 0.1]]}
```

Out[64]=

$$\left\{\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{12}{7} & -\frac{13}{7} & \frac{3400}{7} \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{7} & -\frac{6}{7} & \frac{4800}{7} \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{7} & -\frac{3}{7} & \frac{2400}{7} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 200 \\ 3 & 0 & 0 & 0 & 0 & \frac{13}{7} & \frac{17}{7} & \frac{42\,400}{7} \end{pmatrix}, \begin{pmatrix} -2. & 0. & 0. & 1. & 0. & -1.7 & -1.9 & 485.7 \\ 0. & 0. & 0. & 0. & 1. & -0.7 & -0.9 & 685.7 \\ 1. & 1. & 0. & 0. & 0. & 0.1 & -0.4 & 342.9 \\ 0. & 0. & 1. & 0. & 0. & 0. & 1. & 200. \\ 3. & 0. & 0. & 0. & 0. & 1.9 & 2.4 & 6057.1 \end{pmatrix}\right\}$$

In[44]:=
```
Mod[2400, 7]
```

Out[44]=

6

In[75]:=
```
MatrixForm[A4a = Transpose[Insert[Transpose[A3], {0, 0, 0, 0, 0}, 8]]];
f = 6/7; (*x2,3rd row*)
MatrixForm[A4 = Insert[A4a, {0, 0, 0, 0, 0, -1/7, -4/7, 1, -f}, 5]]
```

Out[77]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{12}{7} & -\frac{13}{7} & 0 & \frac{3400}{7} \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{7} & -\frac{6}{7} & 0 & \frac{4800}{7} \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{7} & -\frac{3}{7} & 0 & \frac{2400}{7} \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 200 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{7} & -\frac{4}{7} & 1 & -\frac{6}{7} \\ 3 & 0 & 0 & 0 & 0 & \frac{13}{7} & \frac{17}{7} & 0 & \frac{42\,400}{7} \end{pmatrix}$$

In[71]:=
```
{MatrixForm[A5 = Pivot[A4, 5, 7]],
Round[A5, 0.1] // MatrixForm}
```

Out[71]=

$$\left\{\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{5}{4} & 0 & -\frac{13}{4} & \frac{977}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{3}{2} & 687 \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{3}{4} & \frac{687}{2} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} & 0 & \frac{7}{4} & \frac{397}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 1 & -\frac{7}{4} & \frac{3}{2} \\ 3 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & \frac{17}{4} & \frac{12\,107}{2} \end{pmatrix}, \begin{pmatrix} -2. & 0. & 0. & 1. & 0. & -1.2 & 0. & -3.2 & 488.5 \\ 0. & 0. & 0. & 0. & 1. & -0.5 & 0. & -1.5 & 687. \\ 1. & 1. & 0. & 0. & 0. & 0.2 & 0. & -0.8 & 343.5 \\ 0. & 0. & 1. & 0. & 0. & -0.2 & 0. & 1.8 & 198.5 \\ 0. & 0. & 0. & 0. & 0. & 0.2 & 1. & -1.8 & 1.5 \\ 3. & 0. & 0. & 0. & 0. & 1.2 & 0. & 4.2 & 6053.5 \end{pmatrix}\right\}$$

In[78]:=
```
MatrixForm[A6a = Transpose[Insert[Transpose[A5], {0, 0, 0, 0, 0, 0}, 9]]];
f = 1/2; (*x4, 1st row*)
MatrixForm[A6 = Insert[A6a, {0, 0, 0, 0, 0, -f/(f - 1)*(3/4 - 1), 0, f/(f - 1)*13/4, 1, -f}, 6]]
```

Out[80]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -\frac{5}{4} & 0 & -\frac{13}{4} & 0 & \frac{977}{2} \\ 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 & -\frac{3}{2} & 0 & 687 \\ 1 & 1 & 0 & 0 & 0 & \frac{1}{4} & 0 & -\frac{3}{4} & 0 & \frac{687}{2} \\ 0 & 0 & 1 & 0 & 0 & -\frac{1}{4} & 0 & \frac{7}{4} & 0 & \frac{397}{2} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 1 & -\frac{7}{4} & 0 & \frac{3}{2} \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{4} & 0 & -\frac{13}{4} & 1 & -\frac{1}{2} \\ 3 & 0 & 0 & 0 & 0 & \frac{5}{4} & 0 & \frac{17}{4} & 0 & \frac{12\,107}{2} \end{pmatrix}$$

In[82]:=
```
{MatrixForm[A7 = Pivot[A6, 6, 8]],
 Round[A7, 0.1] // MatrixForm}
```

Out[82]=

$$\left\{\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 489 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{13} & 0 & 0 & -\frac{6}{13} & \frac{8934}{13} \\ 1 & 1 & 0 & 0 & 0 & \frac{4}{13} & 0 & 0 & -\frac{3}{13} & \frac{4467}{13} \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{13} & 0 & 0 & \frac{7}{13} & \frac{2577}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} & 1 & 0 & -\frac{7}{13} & \frac{23}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{13} & 0 & 1 & -\frac{4}{13} & \frac{2}{13} \\ 3 & 0 & 0 & 0 & 0 & \frac{12}{13} & 0 & 0 & \frac{17}{13} & \frac{78\,687}{13} \end{pmatrix}, \begin{pmatrix} -2. & 0. & 0. & 1. & 0. & -1. & 0. & 0. & -1. & 489. \\ 0. & 0. & 0. & 0. & 1. & -0.4 & 0. & 0. & -0.5 & 687.2 \\ 1. & 1. & 0. & 0. & 0. & 0.3 & 0. & 0. & -0.2 & 343.6 \\ 0. & 0. & 1. & 0. & 0. & -0.4 & 0. & 0. & 0.5 & 198.2 \\ 0. & 0. & 0. & 0. & 0. & 0.4 & 1. & 0. & -0.5 & 1.8 \\ 0. & 0. & 0. & 0. & 0. & 0.1 & 0. & 1. & -0.3 & 0.2 \\ 3. & 0. & 0. & 0. & 0. & 0.9 & 0. & 0. & 1.3 & 6052.8 \end{pmatrix}\right\}$$

In[53]:=
```
Mod[4467, 13]
```

Out[53]=

8

In[83]:=
```
MatrixForm[A8a = Transpose[Insert[Transpose[A7], {0, 0, 0, 0, 0, 0, 0}, 10]];
f = 8/13; (*x2 3rd row*)
MatrixForm[A8 = Insert[A8a, {0, 0, 0, 0, 0, -4/13, 0, 0, f/(f - 1)*(3/13), 1, -f}, 7]]
```

Out[85]//MatrixForm=

$$\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 489 \\ 0 & 0 & 0 & 0 & 1 & -\frac{5}{13} & 0 & 0 & -\frac{6}{13} & 0 & \frac{8934}{13} \\ 1 & 1 & 0 & 0 & 0 & \frac{4}{13} & 0 & 0 & -\frac{3}{13} & 0 & \frac{4467}{13} \\ 0 & 0 & 1 & 0 & 0 & -\frac{5}{13} & 0 & 0 & \frac{7}{13} & 0 & \frac{2577}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{5}{13} & 1 & 0 & -\frac{7}{13} & 0 & \frac{23}{13} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{13} & 0 & 1 & -\frac{4}{13} & 0 & \frac{2}{13} \\ 0 & 0 & 0 & 0 & 0 & -\frac{4}{13} & 0 & 0 & -\frac{24}{65} & 1 & -\frac{8}{13} \\ 3 & 0 & 0 & 0 & 0 & \frac{12}{13} & 0 & 0 & \frac{17}{13} & 0 & \frac{78\,687}{13} \end{pmatrix}$$

In[86]:=
```
{MatrixForm[A9 = Pivot[A8, 7, 6]],
 Round[A9, 0.01] // MatrixForm}
```

Out[86]=

$$\left\{\begin{pmatrix} -2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{1}{5} & -\frac{13}{4} & 491 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\frac{5}{4} & 688 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{3}{5} & 1 & 343 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{5}{4} & 199 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \frac{5}{4} & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{6}{5} & -\frac{13}{4} & 2 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{5} & 3 & 6051 \end{pmatrix}, \begin{pmatrix} -2. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0.2 & -3.25 & 491. \\ 0. & 0. & 0. & 0. & 1. & 0. & 0. & 0. & 0. & -1.25 & 688. \\ 1. & 1. & 0. & 0. & 0. & 0. & 0. & 0. & -0.6 & 1. & 343. \\ 0. & 0. & 1. & 0. & 0. & 0. & 0. & 0. & 1. & -1.25 & 199. \\ 0. & 0. & 0. & 0. & 0. & 0. & 1. & 0. & -1. & 1.25 & 1. \\ 0. & 0. & 0. & 0. & 0. & 0. & 0. & 1. & -0.4 & 0.25 & 0. \\ 0. & 0. & 0. & 0. & 0. & 1. & 0. & 0. & 1.2 & -3.25 & 2. \\ 3. & 0. & 0. & 0. & 0. & 0. & 0. & 0. & 0.2 & 3. & 6051. \end{pmatrix}\right\}$$