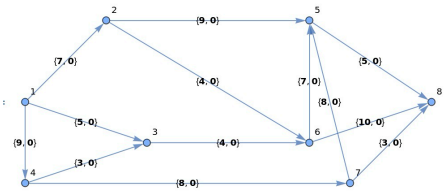
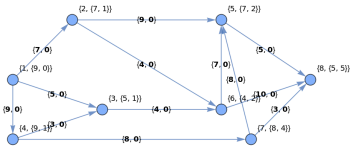
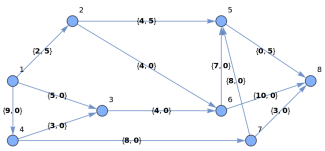
Ex 3:

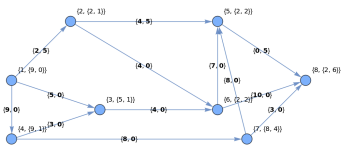




{1, {8, 5, 2, 1}}

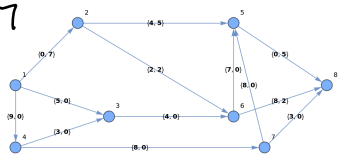
flow: 5

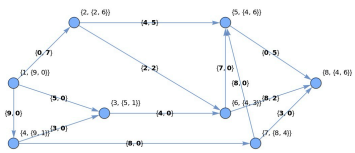




{2, {8, 6, 2, 1}}

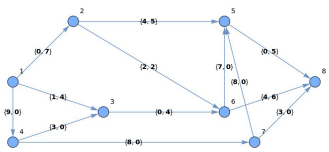
flow: 5+2=7

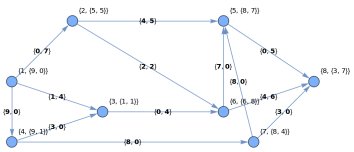




{3, {8, 6, 3, 1}}

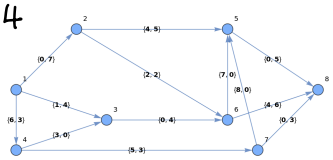
flow 7+4=11

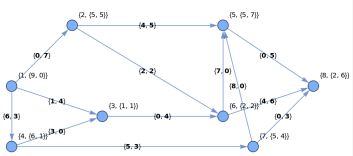




{4, {8, 7, 4, 1}}

flow: 11+3=14

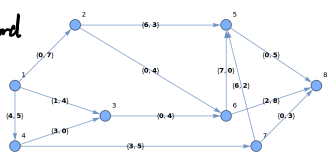




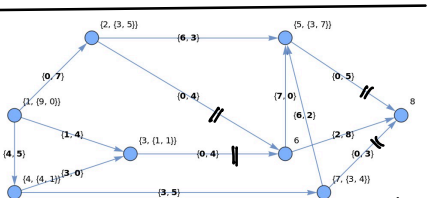
{5, {8, 6, 2, 5, 7, 4, 1}}

path: 1→4→7→5→2→8

backward: 8 ← 6

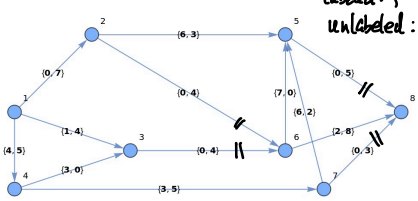


flow: 14+2=16



labeled: {1,2,3,4,5,7}

unlabeled: {6,8}.



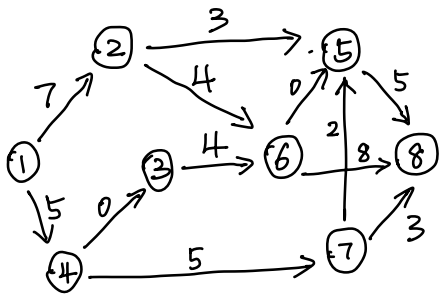
maximal flow = 16

= 7+4+5 (flow out of source)

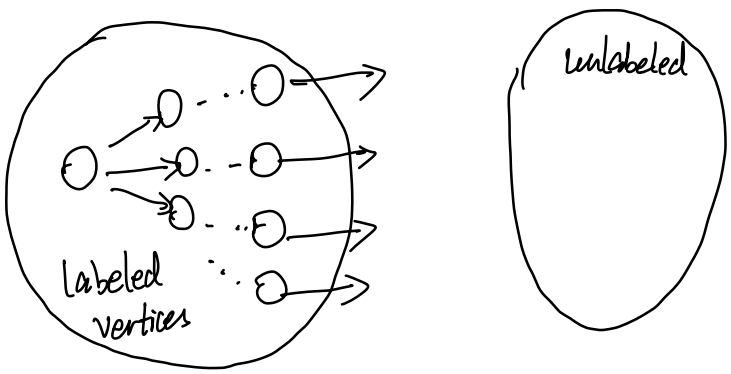
= 5+8+3 (flow into the sink)

= 4+4+5+3 (capacity of the cut)

Max flow:



Thm: Max flow = Min { Capacity of cuts }

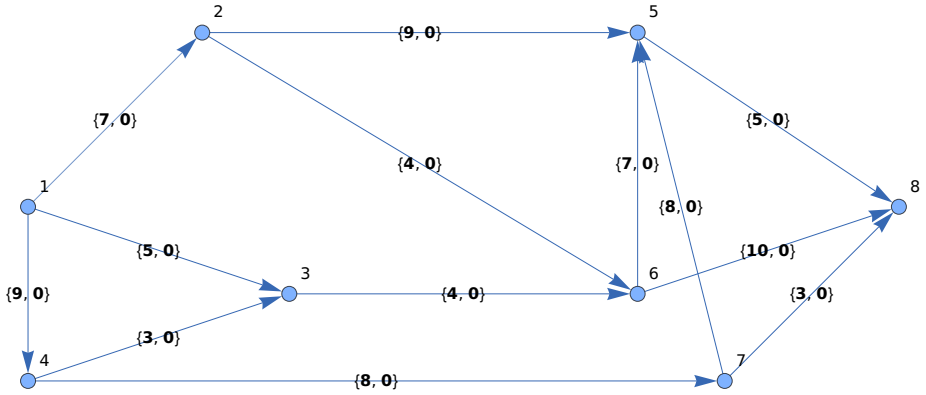


Max flow = Capacity of this last cut.

$\leq$ Capacity of any cut.

In[78]:=
```
EPC = {{{1, 2}, 7}, {{1, 3}, 5}, {{1, 4}, 9}, {{2, 5}, 9}, {{2, 6}, 4}, {{3, 6}, 4}, {{4, 3}, 3}, {{4, 7}, 8},
{{5, 8}, 5}, {{6, 5}, 7}, {{6, 8}, 10}, {{7, 5}, 8}, {{7, 8}, 3}};
Ne = Length[EPC];
El = Table[EPC[[i]][[1, 1]] → EPC[[i]][[1, 2]], {i, 1, Ne}];
Cl1 = Table[{EPC[[i]][[2]], 0}, {i, 1, Ne}];
Lel1 = Table[
Labeled[El[[i]], Cl1[[i]]], {i, 1, Ne}];
VCL = {1 → {-3, 0}, 2 → {-2, 1}, 3 → {-3/2, -1/2},
    4 → {-3, -1}, 5 → {1/2, 1}, 6 → {1/2, -1/2}, 7 → {1, -1}, 8 → {2, 0}};
G0 = Graph[Lel1, VertexLabels → {"Name"}, EdgeLabelStyle → {Bold}, VertexCoordinates → VCL]
Vl = VertexList[G0];
```

Out[84]=



In[86]:=
```
Flow[LVD_, CL_] := Module[{i, j, p, NLVD},
NL = Length[LVD]; (*number of labeled vertices*)
LV = Table[LVD[[i]][[1]], {i, 1, NL}]; (*labeled vertices*)
CL1 = Table[CL[[k]][[1]], {k, 1, Length[CL]}]; (*forward capacity list*)
CL2 = Table[CL[[k]][[2]], {k, 1, Length[CL]}]; (*backward capacity list*)
NLVD = LVD;
For[i = 1, i ≤ NL, i++,
VO = Sort[VertexOutComponent[G0, LV[[i]], {1}]];
VI = Sort[VertexInComponent[G0, LV[[i]], {1}]];
For[j = 1, j ≤ Length[VO], j++,
p = Flatten[Position[El, LV[[i]] → VO[[j]]]][[1]]; (*Position of new edge out of LV[i]*)
If[FreeQ[LV, VO[[j]]] && Cl1[[p]] > 0,
NLVD = Join[NLVD, {{VO[[j]], {Min[Cl1[[p]], LVD[[i, 2]][[1]]], LV[[i]]}}}];
LV = Join[LV, {VO[[j]]}];
];
];
For[j = 1, j ≤ Length[VI], j++,
p = Flatten[Position[El, VI[[j]] → LV[[i]]]][[1]];
If[FreeQ[LV, VI[[j]]] && Cl2[[p]] > 0,
```

```
NLVD = Join[NLVD, {{VI[[j]], {Min[CL2[[p]], LVD[[i, 2]][[2]], LV[[i]]}}}];
LV = Join[LV, {VI[[j]]}]
];
];
];
NLVD
]
Aug[FP_, CL_, d_] := Module[{i, p, NCL},
NCL = CL;
For[i = 1, i ≤ Length[FP] - 1, i++,
If[MemberQ[El, FP[[i + 1]] → FP[[i]]],
p = Flatten[Position[El, FP[[i + 1]] → FP[[i]]]][[1]];
NCL[[p]][[1]] = NCL[[p]][[1]] - d;
NCL[[p]][[2]] = NCL[[p]][[2]] + d,
p = Flatten[Position[El, FP[[i]] → FP[[i + 1]]]][[1]];
NCL[[p]][[1]] = NCL[[p]][[1]] + d;
NCL[[p]][[2]] = NCL[[p]][[2]] - d;
]
];
NCL
]
Vld[VLD_] := Module[{i, VLL, UVL, V1, V2},
VLL = Table[VLD[[i]][[1]], {i, Length[VLD]}];
UVL = Complement[Vl, VLL];
V1 = Table[VLD[[i]][[1]] → ToString[{VLD[[i]][[1]], VLD[[i]][[2]]}], {i, 1, Length[VLD]}];
V2 = Table[UVL[[j]] → "Name", {j, 1, Length[UVL]}];
Join[V1, V2]
]
```

In[102]:=

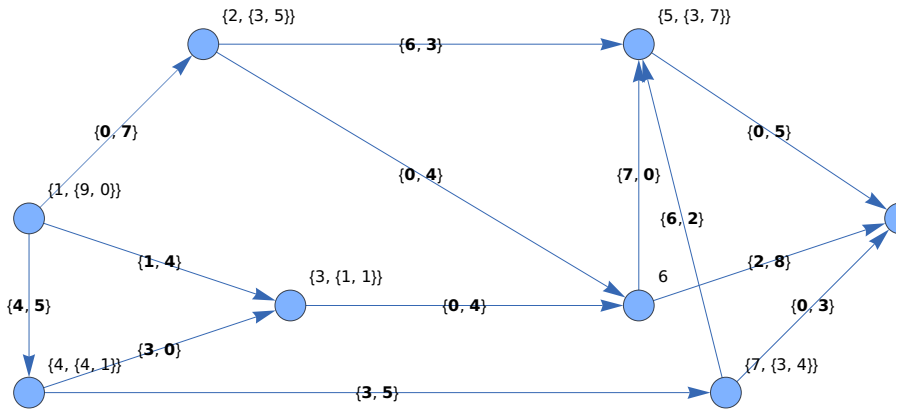
```
N11 = {{1, {9, 0}}};
Cl1 = Table[{EPC[[i]][[2]], 0}, {i, 1, Ne}]; Nls = 0;
Lel1 = Table[
Labeled[El[[i]], Cl1[[i]]], {i, 1, Ne}];


(*run this 6 times to get the maximal flow*)
```

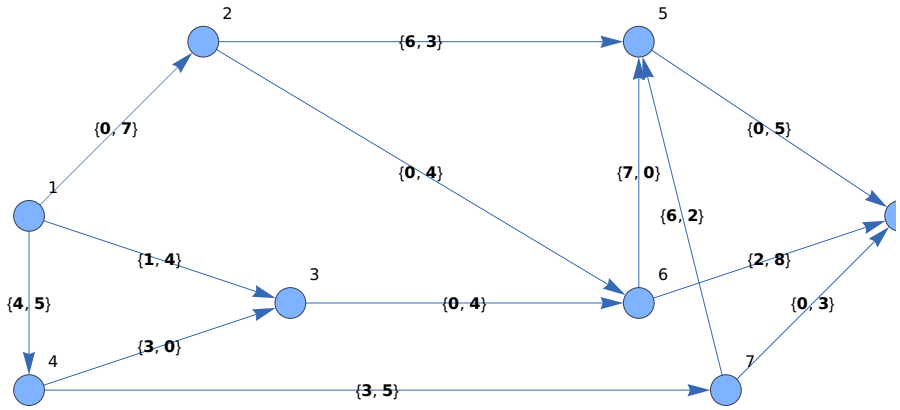
In[175]:=

```
N11 = {{1, {9, 0}}};
N12 = Flow[N11, Cl1] ; (*new labeled vertex list*)
While[Length[N12] > Length[N11],
N11 = N12;
N12 = Flow[N11, Cl1];
]
Vll1 = Vld[N12];
Graph[Lel1, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll1, VertexSize → 0.25]
Lvl = Table[N12[[i]][1], {i, 1, Length[N12]}];
If[MemberQ[Lvl, 8],
Fp1 = {8}; s = 8;
SP = Flatten[Position[Lvl, 8]][1];
FI = N12[SP][2][1];
Nls = Nls + 1;
While[s > 1,
PLV = Flatten[Position[Lvl, s]][1];
s = N12[PLV][2][2];
Fp1 = Join[Fp1, {s}];
];
Cl1 = Aug[Fp1, Cl1, FI];
]
{Nls, Fp1};
Lel1 = Table[
Labeled[El[[i]], Cl1[[i]]], {i, 1, Length[El]}];
Graph[Lel1, VertexLabels → {"Name"}, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll1, VertexSize → 0.25]
```

Out[179]=



Out[184]=



In[185]:=

```
(* Run the following code for step by step labeling process*)
N11 = {{1, {9, 0}}, {2, {7, 1}}, {3, {5, 1}}, {4, {9, 1}}};
Cl1 = Table[{EPC[[i]][[2]], 0}, {i, 1, Ne}]; Nls = 0;
Lel1 = Table[
Labeled[El[[i]], Cl1[[i]], {i, 1, Ne}];
```

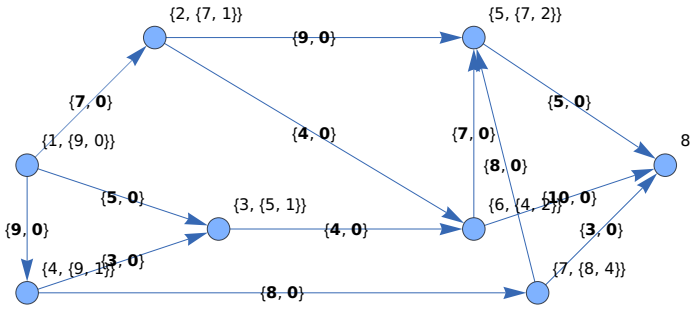
In[188]:=

```
N12 = Flow[N11, Cl1] (*new labeled vertex list*)
(*Vll1=Table[N12[[i]][[1]]→ToString[{N12[[i]][[1]],N12[[i]][[2]]}],{i,1,Length[N12]}];*)
Vll1 = Vld[N12];
Graph[Lel1, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll1, VertexSize → 0.25]
N11 = N12;
```

Out[188]=

{{1, {9, 0}}, {2, {7, 1}}, {3, {5, 1}}, {4, {9, 1}}, {5, {7, 2}}, {6, {4, 2}}, {7, {8, 4}}}

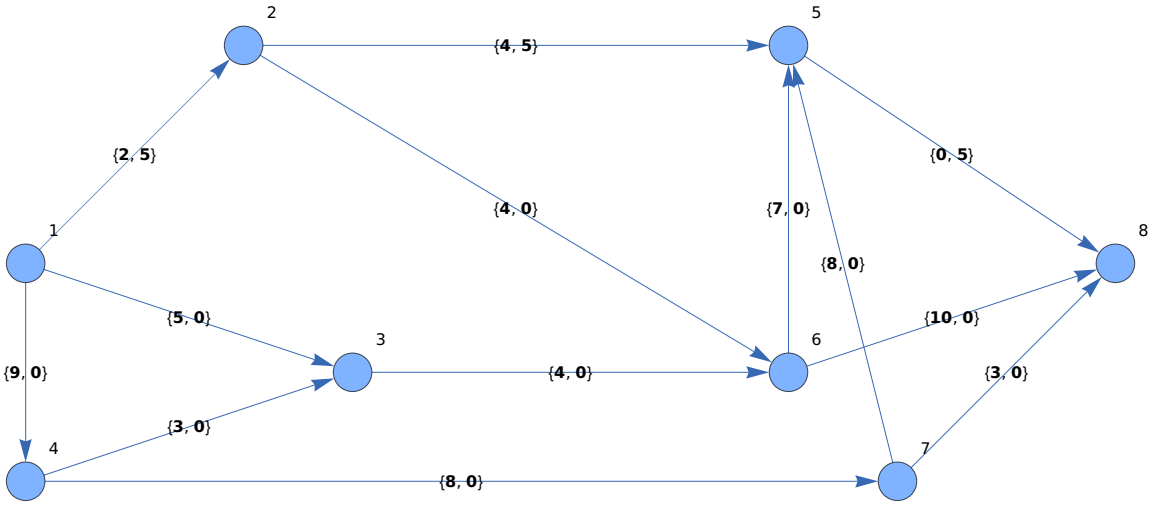
Out[190]=



In[192]:=

```
Fp1 = {8, 5, 2, 1};
Cl2 = Aug[Fp1, Cl1, 5];
Lel2 = Table[
Labeled[El[[i]], Cl2[[i]]], {i, 1, Length[El]}];
Graph[Lel2, VertexLabels → {"Name"}, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll1, VertexSize → 0.25]
```

Out[195]=



In[283]:=

```
N21 = {{1, {9, 0}}, {2, {2, 1}}, {3, {5, 1}}, {4, {9, 1}}};
```

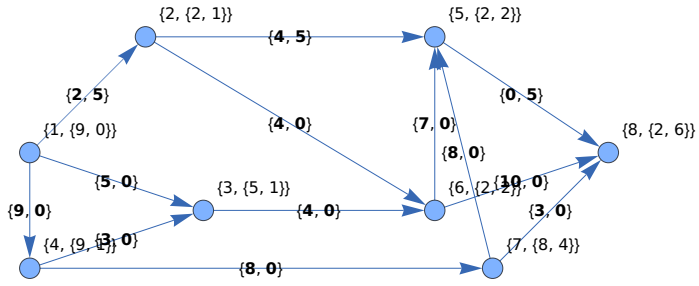
In[288]:=

```
(*Run this twice*)
N22 = Flow[N21, Cl2]
Vll2 = Vld[N22];
Graph[Lel2, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll2, VertexSize → 0.25]
N21 = N22;
```

Out[288]=

```
{{1, {9, 0}}, {2, {2, 1}}, {3, {5, 1}}, {4, {9, 1}}, {5, {2, 2}}, {6, {2, 2}}, {7, {8, 4}}, {8, {2, 6}}}
```

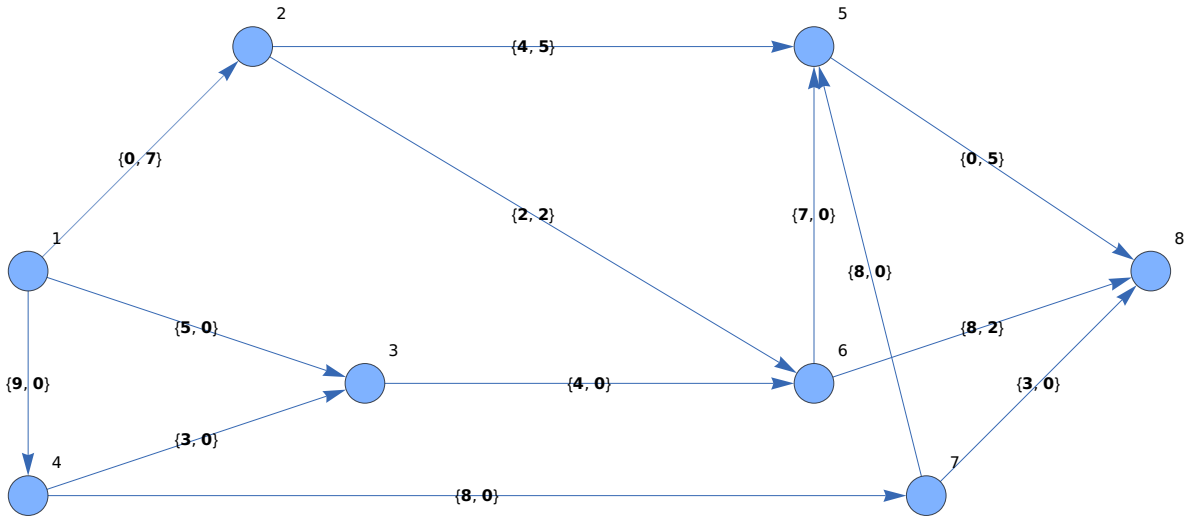
Out[290]=



In[209]:=

```
Fp2 = {8, 6, 2, 1};
Cl3 = Aug[Fp2, Cl2, 2];
Lel3 = Table[
Labeled[El[[i]], Cl3[[i]]], {i, 1, Length[El]}];
Graph[Lel3, VertexLabels → {"Name"}, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll, VertexSize → 0.25]
```

Out[212]=



In[231]:=

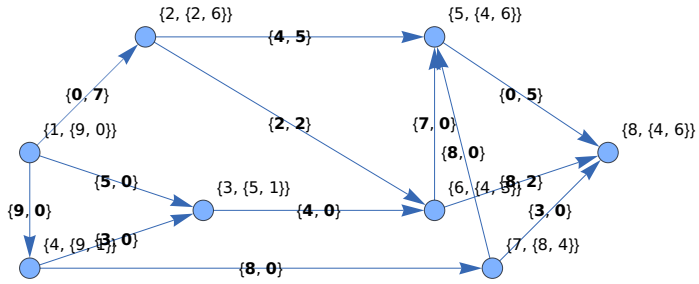
```
N31 = {{1, {9, 0}}, {3, {5, 1}}, {4, {9, 1}}};

(*Run twice*)
N32 = Flow[N31, Cl3]
Vll3 = Vld[N32];
Graph[Lel3, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll3, VertexSize → 0.25]
N31 = N32;
```

Out[236]=

```
{{1, {9, 0}}, {3, {5, 1}}, {4, {9, 1}}, {6, {4, 3}}, {7, {8, 4}}, {5, {4, 6}}, {8, {4, 6}}, {2, {2, 6}}}
```

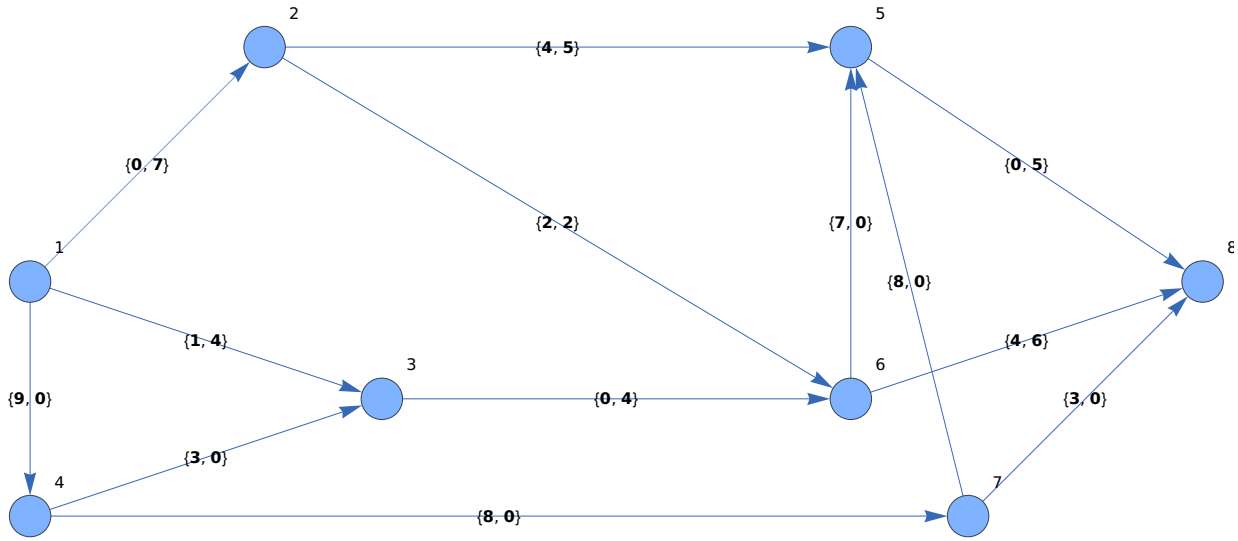
Out[238]=



In[240]:=

```
Fp3 = {8, 6, 3, 1};
Cl4 = Aug[Fp3, Cl3, 4];
Lel4 = Table[
Labeled[El[[i]], Cl4[[i]]], {i, 1, Length[El]}];
G3b = Graph[Lel4, VertexLabels → {"Name"}, EdgeLabelStyle → {Bold},
  VertexCoordinates → VCL, VertexLabels → Vll, VertexSize → 0.25]
```

Out[243]=



In[297]:=

```
N41 = {{1, {9, 0}}, {3, {1, 1}}, {4, {9, 1}}};
```

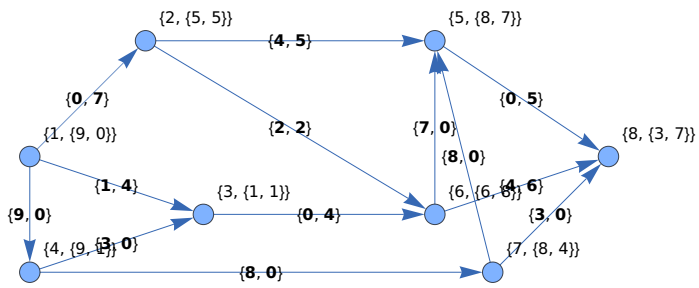
In[310]:=

```
(*Run twice*)
N42 = Flow[N41, Cl4]
Vll4 = Vld[N42];
G4a = Graph[Lel4, EdgeLabelStyle → {Bold},
  VertexCoordinates → VCL, VertexLabels → Vll4, VertexSize → 0.25]
N41 = N42;
```

Out[310]=

```
{{1, {9, 0}}, {3, {1, 1}}, {4, {9, 1}}, {7, {8, 4}}, {5, {8, 7}}, {8, {3, 7}}, {2, {5, 5}}, {6, {6, 8}}}
```

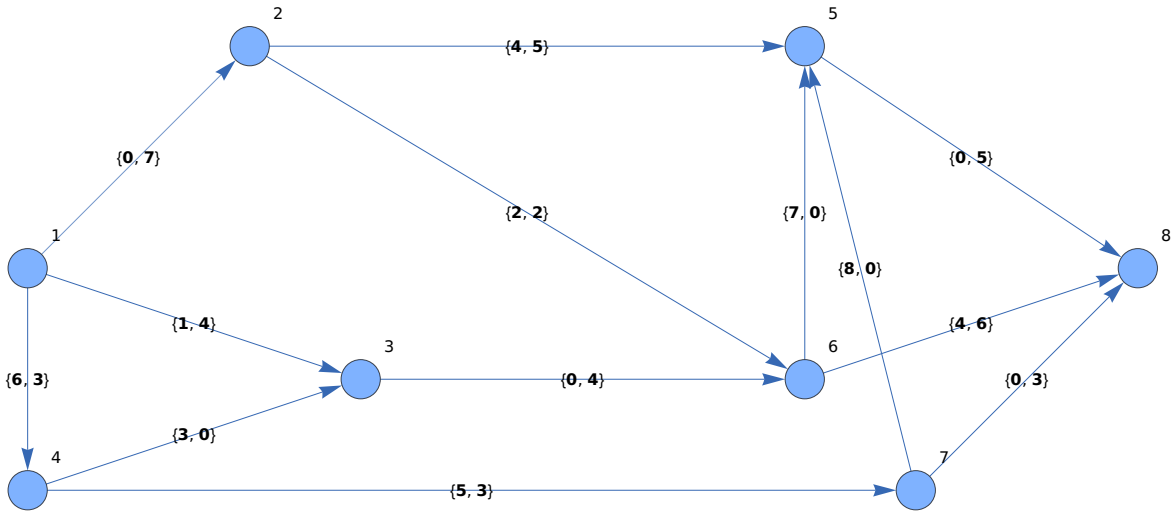
Out[312]=



In[314]:=

```
Fp4 = {8, 7, 4, 1};
Cl5 = Aug[Fp4, Cl4, 3];
Lel5 = Table[
Labeled[El[[i]], Cl5[[i]]], {i, 1, Length[El]}];
Graph[Lel5, VertexLabels → {"Name"}, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll, VertexSize → 0.25]
```

Out[317]=



In[318]:=

```
N51 = {{1, {9, 0}}};
```

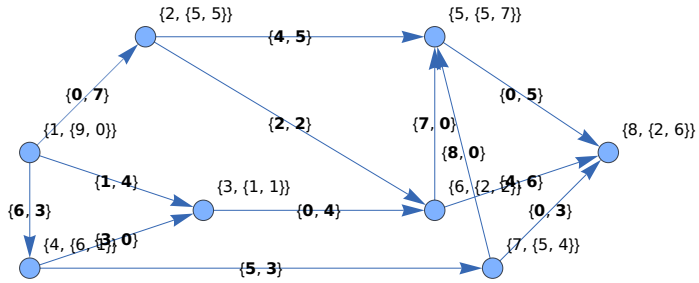
In[343]:=

```
(*Run several times*)
N52 = Flow[N51, Cl5]
Vll5 = Vld[N52];
Graph[Lel5, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll5, VertexSize → 0.25]
N51 = N52;
```

Out[343]=

```
{{1, {9, 0}}, {3, {1, 1}}, {4, {6, 1}}, {7, {5, 4}}, {5, {5, 7}}, {2, {5, 5}}, {6, {2, 2}}, {8, {2, 6}}}
```

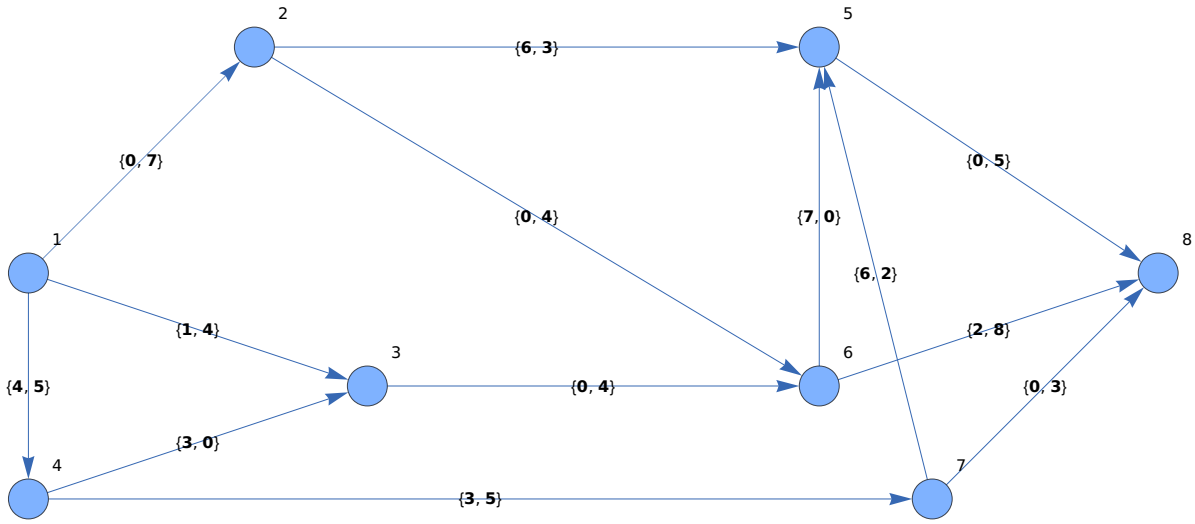
Out[345]=



In[347]:=

```
Fp5 = {8, 6, 2, 5, 7, 4, 1};
Cl6 = Aug[Fp5, Cl5, 2];
Lel6 = Table[
Labeled[El[[i]], Cl6[[i]]], {i, 1, Length[El]}];
Graph[Lel6, VertexLabels → {"Name"}, EdgeLabelStyle → {Bold},
 VertexCoordinates → VCL, VertexLabels → Vll, VertexSize → 0.25]
```

Out[350]=



In[351]:=

```
N61 = {{1, {9, 0}}, {3, {1, 1}}, {4, {4, 1}}};
```

In[368]:=

```
N62 = Flow[N61, Cl6]
Vll6 = Vld[N62];
G4a = Graph[Lel6, EdgeLabelStyle → {Bold},
   VertexCoordinates → VCL, VertexLabels → Vll6, VertexSize → 0.25]
N61 = N62;
```

Out[368]=

{{1, {9, 0}}, {3, {1, 1}}, {4, {4, 1}}, {7, {3, 4}}, {5, {3, 7}}, {2, {3, 5}}}

Out[370]=