Maximize $z = 2x_1 + x_2 + 3x_3$

subj. to $4x_1 + 3x_2 - 3x_2 \le 6$

$2x_1 + 3x_2 + 3x_3 \le 4$

$x_i \ge 0$. x_1, x_3 integers.

$$
\begin{array}{c}
\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array} \\
\left(\begin{array}{ccccc|c}
6 & 6 & 0 & 1 & 1 & 10 \\
\frac{2}{3} & 1 & 1 & 0 & \frac{1}{3} & \frac{4}{3} \\
\hline
0 & 2 & 0 & 0 & 1 & 4
\end{array}\right)
\end{array}
\qquad \frac{4}{3} = 1 + \frac{1}{3} \Rightarrow f = \frac{1}{3}
$$

$$\underset{g_{21}}{\frac{2}{3}} > \underset{f_2}{\frac{1}{3}} \Rightarrow d = \frac{f}{f-1}(g-1)$$

$$= \frac{\frac{1}{3}}{\frac{1}{3}-1} \cdot \left(\frac{2}{3}-1\right) = \frac{\frac{1}{3}}{\frac{2}{3}} \cdot \frac{1}{3} = \frac{1}{6}$$

$$\Rightarrow -\frac{1}{6}x_1 - x_2 - \frac{1}{3}x_5 + x_6 = -\frac{1}{3}$$

$$
\rightsquigarrow
\begin{pmatrix}
6 & 6 & 0 & 1 & 1 & 0 & 10 \\
\frac{2}{3} & 1 & 1 & 0 & \frac{1}{3} & 0 & \frac{4}{3} \\
-\frac{1}{6} & -1 & 0 & 0 & -\frac{1}{3} & 1 & -\frac{1}{3} \\
0 & 2 & 0 & 0 & 1 & 0 & 4
\end{pmatrix}
$$

$$\Downarrow$$

$$
\begin{array}{c|c}
 & \begin{array}{cccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{array} \\
\hline
\begin{array}{c} x_2 \\ x_3 \\ x_1 \\ \\ \end{array} &
\left(\begin{array}{cccccc|c}
0 & 1 & 0 & -\frac{1}{30} & \frac{11}{30} & -\frac{6}{5} & \frac{1}{15} \\
0 & 0 & 1 & -\frac{1}{10} & \frac{1}{10} & \frac{2}{5} & \frac{1}{5} \\
1 & 0 & 0 & \frac{1}{5} & -\frac{1}{5} & \frac{6}{5} & \frac{8}{5} \\
\hline
0 & 0 & 0 & \frac{1}{15} & \frac{4}{15} & \frac{12}{5} & \frac{58}{15}
\end{array}\right)
\end{array}
\qquad \frac{8}{5} = 1 + \underset{f}{\frac{3}{5}}
$$

$$\frac{1}{5} \to \frac{1}{5}$$

$$\rightsquigarrow -\frac{1}{5} \to \frac{\frac{3}{5}}{\frac{3}{5}-1} \cdot \left(-\frac{1}{5}\right) = \frac{3}{-2} \cdot \left(-\frac{1}{5}\right) = \frac{3}{10}$$

$$\frac{6}{5} \to \frac{6}{5}$$

$$\rightsquigarrow -\frac{1}{5}x_4 - \frac{3}{10}x_5 - \frac{6}{5}x_6 + x_7 = -\frac{3}{5}$$

$$\begin{pmatrix} 0 & 1 & 0 & -\frac{1}{30} & \frac{11}{30} & -\frac{6}{5} & 0 & \frac{1}{15} \\ 0 & 0 & 1 & -\frac{1}{10} & \frac{1}{10} & \frac{2}{5} & 0 & \frac{1}{5} \\ 1 & 0 & 0 & \frac{1}{5} & -\frac{1}{5} & \frac{6}{5} & 0 & \frac{8}{5} \\ 0 & 0 & 0 & -\frac{1}{5} & -\frac{3}{10} & -\frac{6}{5} & 1 & -\frac{3}{5} \\ 0 & 0 & 0 & \frac{1}{15} & \frac{4}{15} & \frac{12}{5} & 0 & \frac{58}{15} \end{pmatrix}$$

$$\begin{array}{c c} & \begin{array}{ccccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & \end{array} \\ \begin{array}{c} x_2 \\ x_3 \\ x_1 \\ x_4 \\ \\ \end{array} & \left(\begin{array}{ccccccc|c} 0 & 1 & 0 & 0 & \frac{5}{12} & -1 & -\frac{1}{6} & \frac{1}{6} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 1 & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{2} & 6 & -5 & 3 \\ \hline 0 & 0 & 0 & 0 & \frac{1}{6} & 2 & \frac{1}{3} & \frac{11}{3} \end{array}\right) \end{array}$$

$\frac{1}{2} = f$

$\frac{1}{4} \to \frac{1}{4}$

$1 \longrightarrow 1$

$-\frac{1}{2} \mapsto \frac{\frac{1}{2}}{\frac{1}{2}-1} \cdot \left(-\frac{1}{2}\right) = \frac{1}{2}$

$\rightsquigarrow \; -\frac{1}{4}x_5 - x_6 - \frac{1}{2}x_7 + x_8 = -\frac{1}{2}$

$$\begin{pmatrix} 0 & 1 & 0 & 0 & \frac{5}{12} & -1 & -\frac{1}{6} & 0 & \frac{1}{6} \\ 0 & 0 & 1 & 0 & \frac{1}{4} & 1 & -\frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & \frac{3}{2} & 6 & -5 & 0 & 3 \\ 0 & 0 & 0 & 0 & -\frac{1}{4} & -1 & -\frac{1}{2} & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & \frac{1}{6} & 2 & \frac{1}{3} & 0 & \frac{11}{3} \end{pmatrix} \rightsquigarrow$$

$x_2 \; x_3 \; x_4 \; x_5 \; x_6 \; x_7 \; x_8$

$$\begin{array}{c c} \begin{array}{c} \\ x_3 \\ x_1 \\ \\ \\ \\ \end{array} & \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \frac{8}{3} & 1 & -\frac{5}{3} & \frac{2}{3} \\ 0 & -1 & 1 & 0 & 0 & \frac{8}{3} & 0 & -\frac{2}{3} & \frac{2}{3} \\ 1 & 2 & 0 & 0 & 0 & -\frac{10}{3} & 0 & \frac{4}{3} & \frac{2}{3} \\ 0 & -8 & 0 & 1 & 0 & \frac{64}{3} & 0 & -\frac{22}{3} & \frac{16}{3} \\ 0 & 2 & 0 & 0 & 1 & -\frac{4}{3} & 0 & -\frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 0 & \frac{2}{3} & \frac{10}{3} \end{pmatrix} \end{array}$$

$\frac{2}{3} = f$

$-1 \to \frac{\frac{2}{3}}{\frac{2}{3}-1}(-1) = 2$

$\frac{8}{3} \to \frac{8}{3}$

$-\frac{2}{3} \to \frac{\frac{2}{3}}{\frac{2}{3}-1} \cdot \left(-\frac{2}{3}\right) = \frac{4}{3}$

$\rightsquigarrow \; -2x_2 - \frac{8}{3}x_6 - \frac{4}{3}x_8 + x_9 = -\frac{2}{3}$

$$\begin{pmatrix} 0 & -1 & 0 & 0 & 0 & \frac{8}{3} & 1 & -\frac{5}{3} & 0 & \frac{2}{3} \\ 0 & -1 & 1 & 0 & 0 & \frac{8}{3} & 0 & -\frac{2}{3} & 0 & \frac{2}{3} \\ 1 & 2 & 0 & 0 & 0 & -\frac{10}{3} & 0 & \frac{4}{3} & 0 & \frac{2}{3} \\ 0 & -8 & 0 & 1 & 0 & \frac{64}{3} & 0 & -\frac{22}{3} & 0 & \frac{16}{3} \\ 0 & 2 & 0 & 0 & 1 & -\frac{4}{3} & 0 & -\frac{2}{3} & 0 & \frac{2}{3} \\ 0 & -2 & 0 & 0 & 0 & -\frac{8}{3} & 0 & -\frac{4}{3} & 1 & -\frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 0 & \frac{2}{3} & 0 & \frac{10}{3} \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 4 & 1 & -1 & -\frac{1}{2} & 1 \\ 0 & 0 & 1 & 0 & 0 & 4 & 0 & 0 & -\frac{1}{2} & 1 \\ 1 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 32 & 0 & -2 & -4 & 8 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{4}{3} & 0 & \frac{2}{3} & -\frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{4}{3} & 0 & \frac{2}{3} & 0 & \frac{10}{3} \end{pmatrix}$$

$\Rightarrow$ optimal solution:

$(x_1, x_2, x_3) = (0, \frac{1}{3}, 1)$

Maximum: $2 \cdot 0 + \frac{1}{3} + 3 \cdot 1 = \frac{10}{3}$

Transportation Problem:

LA	Chicago	NYC	SA		
5	7	9	6	SLC	120 tons
6	7	10	5	Denver	140
7	6	8	1	SA	100
100	60	80	120		

(tons)

$\parallel$ 360 (demand) $\qquad$ $\parallel$ 360 (supply)

Demand $\qquad$ supply

c_{ij}: cost per unit from i-th plant to j-th warehouse

x_{ij}: amount - - - - - - - - - - -

$$z = \sum_{i=1}^{3} \sum_{j=1}^{4} c_{ij} x_{ij}$$

Subject:
$$\sum_{j=1}^{4} x_{ij} \leq s_i, \quad i=1,2,3$$
supply of i-th plant.

$$\sum_{i=1}^{3} x_{ij} \geq d_j, \quad j=1,\cdots,4$$
demand of j-th warehouse.

$x_{ij} \geq 0$, x_{ij} integer.

$=$

$$\sum_{i=1}^{3} s_i = 360 = \sum_{j=1}^{4} d_j.$$

LA	Chicago	NYC	SA	
5	7	9	6	
6	7	10	5	
7	6	8	1	
100	60	80	120	

$\rightsquigarrow$

$0+10-9=1$

$5+(-1)-6 = -2$

$\boxed{5}$ 100	$\boxed{7}$ 0 (1)	$\boxed{9}$ 0	$\boxed{6}$ 20	120	0	v_1
$\boxed{6}$ 0 (−2)	$\boxed{7}$ 40	$\boxed{10}$ 0	$\boxed{5}$ 100	140	−1	v_2
$\boxed{7}$ 0	$\boxed{6}$ 20	$\boxed{8}$ 80	$\boxed{1}$ 0 (3)	100	−2	v_3
100	60	80	120			
5	8	10	6			

$$\begin{cases} v_1 + w_1 = 5 \\ v_1 + w_4 = 6 \\ v_2 + w_2 = 7 \\ v_2 + w_4 = 5 \\ v_3 + w_2 = 6 \\ v_3 + w_3 = 8 \end{cases} \qquad \begin{array}{l} v_1 = 0 \\ \Downarrow \\ w_1 = 5 \\ w_4 = 6 \\ w_2 = 8 \\ v_2 = -1 \\ v_3 = -2 \\ w_3 = 10 \end{array}$$

c_B	$c_1 \; c_2 \; \cdots \; c_n$		
m rows	B		b
	$z_1 \; z_2 \; \cdots \; z_n$		

$\rightsquigarrow$ $B = (A_{i_1} \cdots A_{i_m})$

c_B	c_k	c_j
$c_{i_1}, \ldots, c_{i_m}$	$B^{-1}A_k$	$\boxed{B^{-1}A_j} = t_j$
	$0 \cdots 0$	$c_B \cdot t_j - c_j$

$0 = \boxed{c_B \cdot (B^{-1})_k A_k = c_k}$

$\downarrow$

$A_k^T \cdot (c_B (B^{-1})_k)^T = c_k$