3.6 EXERCISES

1. Consider the linear programming problem

$$\text{Maximize} \quad z = x_1 + 2x_2 + x_3 + x_4$$

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &\le 8 \\ 2x_1 + 3x_2 \qquad + 4x_4 &\le 12 \\ 3x_1 + x_2 + 2x_3 \qquad &\le 18 \\ x_j \ge 0 \quad 1 \le j &\le 4. \end{aligned}$$

$$2x_1 + x_2 + 3x_3 + x_4 + x_5 = 8$$
$$2x_1 + 3x_2 + \quad + 4x_4 + x_6 = 12$$
$$3x_1 + x_2 + 2x_3 \quad + x_7 = 18$$

		1	2	1	1	0	0	0	
c_B		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	x_5	2	1	3	1	1	0	0	8
0	x_6	2	3	0	4	0	1	0	12
0	x_7	3	1	2	0	0	0	1	18
		-1	-2	-1	-1	0	0	0	0

$$\rightarrow B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{9} & 0 \\ 0 & \frac{1}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{9} & 1 \end{pmatrix}$$

After adding slack variables x_5, x_6, and x_7 and solving by the simplex method, we obtain the final tableau shown below.

$2+\Delta c_2 = c_2 + \Delta c_2$ $\qquad$ $c_4 + \Delta c_4$

		1	2	1	$c_4 = 1$	0	0	0	
$\mathbf{c}_B$		$\boxed{x_1}$	x_2	x_3	$\boxed{x_4}$	$\boxed{x_5}$	$\boxed{x_6}$	x_7	$\mathbf{x}_B$
$\rightarrow$ 1	x_3	$\frac{4}{9}$	0	1	$-\frac{1}{9}$	$\frac{1}{3}$	$-\frac{1}{9}$	0	$\frac{4}{3}$
$2+\Delta c_2 \leftarrow$ 2	x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	4
0	x_7	$\frac{13}{9}$	0	0	$-\frac{10}{9}$	$-\frac{2}{3}$	$-\frac{1}{9}$	1	$\frac{34}{3}$
$z_j - c_j$		$\frac{7}{9}$	0	0	$\frac{14}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	0	$\frac{28}{3}$

$z_1 - c_1$ $\qquad$ $\frac{14}{9} - \Delta c_4 \ge 0$

(a) For each of the cost coefficients c_j, $1 \le j \le 4$, find the range of values for Δc_j for which the above solution remains optimal.

(b) For each of the resources b_i, $1 \le i \le 3$, find the range of values for Δb_i for which the above solution remains feasible.

Δc_1: $\quad \Delta c_1 \le z_1 - c_1 = \frac{7}{9}$, $\quad -\infty < \Delta c_1 \le \frac{7}{9}$

Δc_2: $\quad \Delta c_2 \ge \max \left\{ -\frac{z_j - c_j}{t_{2j}} : t_{2j} > 0 \right\}$

j nonbasic.

$$= \max \left\{ -\frac{\frac{7}{\cancel{9}3}}{\frac{2}{\cancel{3}}}, -\frac{\frac{14}{\cancel{9}3}}{\frac{4}{\cancel{3}}}, -\frac{\frac{5}{\cancel{9}3}}{\frac{1}{\cancel{3}}} \right\} = -\frac{7}{6}$$

$$\left\{ -\frac{7}{6}, -\frac{7}{6}, -\frac{5}{3} \right\}$$

$$\Rightarrow -\frac{7}{6} \le \Delta c_2 < +\infty$$

$$\frac{5}{6} = 2 - \frac{7}{6} \le c_2 + \Delta c_2 < +\infty$$

Δc_3: $\quad \Delta c_3 \ge \max \left\{ -\frac{\frac{7}{9}}{\frac{4}{9}}, -\frac{\frac{1}{3}}{\frac{1}{3}} \right\} = -1$

$$-\frac{7}{4} \qquad -1$$

$$\Delta c_3 \le \min \left\{ \frac{\frac{14}{9}}{\frac{1}{9}}, \frac{\frac{5}{9}}{\frac{1}{9}} \right\} = 5.$$

$$14 \qquad 5$$

$$-1 \le \Delta c_3 \le 5$$

$$0 \le c_3 + \Delta c_3 \le 6.$$

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subject to

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After adding slack variables x_5, x_6, and x_7 and solving by the simplex method, we obtain the final tableau shown below.

		1	2	1	1	0	0	0	
$\mathbf{c}_B$		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{x}_B$
1	x_3	$\frac{4}{9}$	0	1	$-\frac{1}{9}$	$\frac{1}{3}$	$-\frac{1}{9}$	0	$\frac{4}{3}$
2	x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	4
0	x_7	$\frac{13}{9}$	0	0	$-\frac{10}{9}$	$-\frac{2}{3}$	$-\frac{1}{9}$	1	$\frac{34}{3}$
		$\frac{7}{9}$	0	0	$\frac{14}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	0	$\frac{28}{3}$

$(B^{-1})_2$

$\Delta b_2 \geq -\frac{4}{\frac{1}{3}} = -12$

$\Delta b_2 \leq \min\left\{\frac{\frac{4}{3}}{\frac{1}{9}}, \frac{\frac{34}{3}}{\frac{1}{9}}\right\}$
$= 12$

$-12 \leq \Delta b_2 \leq 12$

$(B^{-1})_3$:

$\Delta b_3 \geq -\frac{\frac{34}{3}}{1} = -\frac{34}{3}.$

(a) For each of the cost coefficients c_j, $1 \le j \le 4$, find the range of values for Δc_j for which the above solution remains optimal.

(b) For each of the resources b_i, $1 \le i \le 3$, find the range of values for Δb_i for which the above solution remains feasible.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 + x_5 &= 8 \\ 2x_1 + 3x_2 + \quad + 4x_4 + x_6 &= 12 \\ 3x_1 + x_2 + 2x_3 \quad + x_7 &= 18 \end{aligned}$$

		1	2	1	1	0	0	0	
c_B		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	x_5	2	1	3	1	1	0	0	8
0	x_6	2	3	0	4	0	1	0	12
0	x_7	3	1	2	0	0	0	1	18
		-1	-2	-1	-1	0	0	0	0

$$\rightsquigarrow B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{9} & 0 \\ 0 & \frac{1}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{9} & 1 \end{pmatrix}$$

Δb_1: $\quad b_1 \rightsquigarrow b_1 + \Delta b_1$

$$x_B = \underset{\substack{\| \\ \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}}{B^{-1} \cdot b} \rightsquigarrow B^{-1} \cdot \underbrace{\begin{pmatrix} b_1 + \Delta b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{\hat{b}} = \underbrace{B^{-1} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{x_B} + B^{-1} \cdot \begin{pmatrix} \Delta b_1 \\ 0 \\ 0 \end{pmatrix}$$

$$= x_B + (\Delta b_1) \underbrace{\left(B^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)}_{(B^{-1})_1}$$

$$\hat{x}_B = x_B + (\Delta b_1) \cdot (B^{-1})_1$$

$$= \begin{pmatrix} \frac{4}{3} \\ 4 \\ \frac{34}{3} \end{pmatrix} + (\Delta b_1) \cdot \begin{pmatrix} \frac{1}{3} \\ 0 \\ -\frac{2}{3} \end{pmatrix} \geq 0 \Rightarrow \begin{cases} \frac{1}{3} \Delta b_1 \geq -\frac{4}{3} \\ 4 + 0 \cdot b_1 = 4 \geq 0 \\ \frac{2}{3} \Delta b_1 \leq \frac{34}{3} \end{cases}$$

$$\Rightarrow \begin{cases} \Delta b_1 \geq -\frac{\frac{4}{3}}{\frac{1}{3}} = -4 \\ \Delta b_1 \leq \frac{\frac{34}{3}}{\frac{2}{3}} = 17. \end{cases}$$

$-4 \leq \Delta b_1 \leq 17.$

3.6 EXERCISES

1. Consider the linear programming problem

$$\text{Maximize} \quad z = x_1 + 2x_2 + x_3 + x_4$$

subject to

$$\begin{aligned} 2x_1 + x_2 + 3x_3 + x_4 &\le 8 \\ 2x_1 + 3x_2 \qquad + 4x_4 &\le 12 \\ 3x_1 + x_2 + 2x_3 \qquad &\le 18 \\ x_j \ge 0 \quad 1 \le j \le 4. \end{aligned}$$

$\frac{1}{2}$

$\mathbf{c}_B$		1	2	1	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	$\mathbf{x}_B$
1	x_3	$\frac{4}{19}$	0	1	$-\frac{1}{9}$	$\frac{1}{3}$	$-\frac{1}{9}$	0	$\frac{4}{3}$
2	x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	4
0	x_7	$\frac{13}{9}$	0	0	$-\frac{10}{9}$	$-\frac{2}{3}$	$-\frac{1}{9}$	1	$\frac{34}{3}$
		$\frac{7}{9}$	0	0	$\frac{14}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	0	$\frac{28}{3}$

$1-\frac{1}{2}=\frac{1}{2}$

$-\frac{2}{9}$, $\frac{26}{3}$, $\frac{88}{9}$

$z_j - c_j + \Delta c_j \, t_{1j}$ $\qquad$ $\frac{7}{9} - \frac{1}{2}\cdot\frac{4}{9} = \frac{10}{18}$

(a) For each of the cost coefficients c_j, $1 \le j \le 4$, find the range of values for Δc_j for which the above solution remains optimal.

(b) For each of the resources b_i, $1 \le i \le 3$, find the range of values for Δb_i for which the above solution remains feasible.

$0 \le b_2 + \Delta b_2 \le 24$

$-12 \le \Delta b_2 \le 12$

2. What will be an optimal solution to the problem in Exercise 1
   (a) if c_1 is changed to 3?
   (b) if b_2 is changed to 26?
   (c) if c_3 is changed to $\frac{1}{2}$?
   (d) if b_3 is changed to 127?

$\mathbf{x}_B \leftarrow$

$0 \le c_3 \le 6$

$-1 \le \Delta c_3 \le 5$

$$-\infty < \Delta c_1 \le \frac{7}{9} \Rightarrow \quad -\infty < c_1 \le 1 + \frac{7}{9} = \frac{16}{9}$$

$$2x_1 + x_2 + 3x_3 + x_4 + x_5 = 8$$
$$2x_1 + 3x_2 + \quad + 4x_4 + x_6 = 12$$
$$3x_1 + x_2 + 2x_3 \quad + x_7 = 18$$

c_B		1	2	1	1	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	x_7	
0	x_5	2	1	3	1	1	0	0	8
0	x_6	2	3	0	4	0	1	0	12
0	x_7	3	1	2	0	0	0	1	18
		-1	-2	-1	-1	0	0	0	0

$$\hat{x}_B = x_B + \Delta b_2 (B^{-1})_2$$
$$= \begin{pmatrix} \frac{4}{3} \\ 4 \\ \frac{34}{3} \end{pmatrix} + \underset{(26-12)}{\underset{\shortparallel}{14}} \begin{pmatrix} -\frac{1}{9} \\ \frac{1}{3} \\ -\frac{1}{9} \end{pmatrix} = \begin{pmatrix} -\frac{2}{9} \\ \frac{26}{3} \\ \frac{102-14}{9} = \frac{88}{9} \end{pmatrix}$$

$$\rightarrow B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}$$

$$B^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{9} & 0 \\ 0 & \frac{1}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{9} & 1 \end{pmatrix}$$

Ex: Maximize $z = 5x_1 + 6x_2$

subject to $10x_1 + 3x_2 \le 52$

$2x_1 + 3x_2 \le 18$

$x_1 \ge 0,\ x_2 \ge 0$ integers.

$$\begin{cases} 10x_1 + 3x_2 + x_3 = 52 \\ 2x_1 + 3x_2 + x_4 = 18 \end{cases}$$

$$\begin{cases} 10x_1 + 3x_2 = 52 \\ 2x_1 + 3x_2 = 18 \end{cases} \Rightarrow (x_1, x_2) = \left(\frac{17}{4}, \frac{19}{6}\right).$$

basic solution.

		5	6	0	0	
C_B		x_1	x_2	x_3	x_4	x_B
5	x_1	1	0	$\frac{1}{8}$	$-\frac{1}{8}$	$\frac{17}{4}$
6	x_2	0	1	$-\frac{1}{12}$	$\frac{5}{12}$	$\frac{19}{6}$
		0	0	$\frac{1}{8}$	$\frac{15}{8}$	$\frac{161}{4}$

$\frac{17}{4} = 4 + \frac{1}{4} \rightsquigarrow 4$

$\frac{19}{6} = 3 + \frac{1}{6} \rightsquigarrow 3$

$z = 5 \cdot 4 + 6 \cdot 3 = 38.$

$(x_1, x_2) = (3, 4) \Rightarrow z = 5 \cdot 3 + 6 \cdot 4 = 15 + 24 = 39.$

$\underbrace{x_1 + 0 \cdot x_3 - 1 \cdot x_4}_{} \quad (0 = \lfloor \tfrac{1}{8} \rfloor,\ -1 = \lfloor -\tfrac{1}{8} \rfloor)$

$\le$

① $\boxed{x_1 + \frac{1}{8}x_3 - \frac{1}{8}x_4 = \frac{17}{4}}$

$x_2 - \frac{1}{12}x_3 + \frac{5}{12}x_4 = \frac{19}{6}$

$a - 1 < \lfloor a \rfloor \le a$

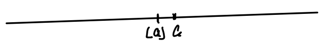
$\lfloor 2 \rfloor = 2$

$\lfloor \frac{7}{2} \rfloor = 3$, $\frac{7}{2} = 3 + \frac{1}{2}$

$\lfloor -\frac{7}{2} \rfloor = -4$, $-\frac{7}{2} = -4 + \frac{1}{2} = -3 - \frac{1}{2}$

$\boxed{x_1 + 0 \cdot x_3 - 1 \cdot x_4 \le \lfloor \tfrac{17}{4} \rfloor = 4.}$

①' $x_1 + 0 \cdot x_3 - 1 \cdot x_4 + u_1 = 4.$ $\quad (u_1 \ge 0)$



①' − ① $\quad \boxed{-\frac{1}{8}x_3 - \frac{7}{8}x_4 + u_1 = -\frac{1}{4}}$

$(x_1, x_2, x_3, x_4) = (*, *, 0, 0)$