3.6 EXERCISES

1. Consider the linear programming problem

$$\text{Maximize} \quad z = x_1 + 2x_2 + x_3 + x_4$$

subject to

$$2x_1 + x_2 + 3x_3 + x_4 \le 8$$
$$2x_1 + 3x_2 + 4x_4 \le 12$$
$$3x_1 + x_2 + 2x_3 \le 18$$
$$x_j \ge 0 \quad 1 \le j \le 4.$$

After adding slack variables x_5, x_6, and x_7 and solving by the simplex method, we obtain the final tableau shown below.

$C_2 + \Delta C_2$

$\mathbf{c}_B$		1 x_1	$C_2=2$ x_2	1 x_3	1 x_4	0 x_5	0 x_6	0 x_7	$\mathbf{x}_B$
1	x_3	$\frac{4}{9}$	0	1	$-\frac{1}{9}$	$\frac{1}{3}$	$-\frac{1}{9}$	0	$\frac{4}{3}$
2	x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	4
0	x_7	$\frac{13}{9}$	0	0	$-\frac{10}{9}$	$-\frac{2}{3}$	$-\frac{1}{9}$	1	$\frac{34}{3}$
		$\frac{7}{9}$	0	0	$\frac{14}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	0	$\frac{28}{3}$

t_1

$\begin{pmatrix} 1 \\ C_2+\Delta C_2 \\ 0 \end{pmatrix}$

$0 \le 1\cdot\frac{4}{9} + (C_2+\Delta C_2)\cdot\frac{2}{3} + 0\cdot\frac{13}{9} - 1 = (C_B^T t_1 - C_1) + \Delta C_2 \frac{2}{3} \ge 0 \Rightarrow \boxed{\Delta C_2 \ge -\frac{1}{\frac{2}{3}}\left(\frac{7}{9}\right) = -\frac{z_1 - c_1}{t_{21}}}$

(a) For each of the cost coefficients c_j, $1 \le j \le 4$, find the range of values for Δc_j for which the above solution remains optimal.

(b) For each of the resources b_i, $1 \le i \le 3$, find the range of values for Δb_i for which the above solution remains feasible.

$2x_1 + x_2 + 3x_3 + x_4 + x_5 = 8$
$2x_1 + 3x_2 + 4x_4 + x_6 = 12$
$3x_1 + x_2 + 2x_3 + x_7 = 18$

$\underbrace{C_B^T \cdot t_1}_{z_1} - (C_1 + \Delta C_1) \ge 0 \Rightarrow \Delta C_1 \le C_B^T \cdot t_1 - C_1 = \frac{7}{9} \Rightarrow C_1 \nearrow \frac{16}{9}$

$C_B^T t_4 - (C_4 + \Delta C_4) \ge 0 \Rightarrow \Delta C_4 \le C_B^T t_4 - C_4 = \frac{14}{9} \Rightarrow C_4 \nearrow \frac{23}{9}$

C_B		1 x_1	2 x_2	1 x_3	1 x_4	0 x_5	0 x_6	0 x_7	
0	x_5	2	1	3	1	1	0	0	8
0	x_6	2	3	0	4	0	1	0	12
0	x_7	3	1	2	0	0	0	1	18
		-1	-2	-1	-1	0	0	0	0

$\rightarrow B = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 0 \\ 2 & 1 & 1 \end{pmatrix}$

$B^{-1} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{9} & 0 \\ 0 & \frac{1}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{9} & 1 \end{pmatrix}$

$\Delta C_2 \ge -\frac{\frac{5}{9}}{\frac{1}{3}} = -\frac{5}{3}$

$C_2 + \Delta C_2$

$\mathbf{c}_B$		1 x_1	$C_2=2$ x_2	1 x_3	1 x_4	0 x_5	0 x_6	0 x_7	$\mathbf{x}_B$
1	x_3	$\frac{4}{9}$	0	1	$-\frac{1}{9}$	$\frac{1}{3}$	$-\frac{1}{9}$	0	$\frac{4}{3}$
2	x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	4
0	x_7	$\frac{13}{9}$	0	0	$-\frac{10}{9}$	$-\frac{2}{3}$	$-\frac{1}{9}$	1	$\frac{34}{3}$
		$\frac{7}{9}$	0	0	$\frac{14}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	0	$\frac{28}{3}$

t_4

$C_3 = 1$
$2 + \Delta C_2 = C_2 + \Delta C_2$
$C_7 = 0$

$\Delta C_2 \ge -\frac{1}{\frac{2}{3}}\left(\frac{7}{9}\right) = -\frac{7}{6}$

$\underbrace{C_B^T \cdot t_4 + \frac{4}{3}\cdot\Delta C_2 - 1 \ge 0}$
$\frac{14}{9} + \frac{4}{3}\Delta C_2 \ge 0$

$\Rightarrow \Delta C_2 \ge -\frac{\frac{14}{9}}{\frac{4}{3}} = -\frac{7}{6}$

$\Rightarrow \Delta C_2 \ge \max\left\{-\frac{7}{6}, -\frac{7}{6}, -\frac{5}{3}\right\} = -\frac{7}{6}$.

$\left(\Delta C_{i_r} \ge \max\left\{ -\frac{1}{t_{rj}}(z_j - c_j),\ j \text{ non-basic},\ t_{rj} > 0 \right\} \right)$

$\Delta C_3 \ge -\frac{7}{4}$ $\quad$ $\Delta C_3 \le 14$ $\quad$ $\Delta C_3 \ge -\frac{\frac{1}{3}}{\frac{1}{3}} = -1$ $\quad$ $\Rightarrow \Delta C_3 \le \frac{\frac{5}{9}}{\frac{1}{9}} = 5$

$C_3 + \Delta C_3$

$\mathbf{c}_B$		1 x_1	2 x_2	$C_3=1$ x_3	1 x_4	0 x_5	0 x_6	0 x_7	$\mathbf{x}_B$
1	x_3	$\frac{4}{9}$	0	1	$-\frac{1}{9}$	$\frac{1}{3}$	$-\frac{1}{9}$	0	$\frac{4}{3}$
2	x_2	$\frac{2}{3}$	1	0	$\frac{4}{3}$	0	$\frac{1}{3}$	0	4
0	x_7	$\frac{13}{9}$	0	0	$-\frac{10}{9}$	$-\frac{2}{3}$	$-\frac{1}{9}$	1	$\frac{34}{3}$
		$\frac{7}{9}$	0	0	$\frac{14}{9}$	$\frac{1}{3}$	$\frac{5}{9}$	0	$\frac{28}{3}$

t_4

$\begin{pmatrix} 1+\Delta C_3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} C_3 + \Delta C_3 \\ C_2 \\ C_7 \end{pmatrix}$

$\Delta C_3 \ge -\frac{1}{t_{31}}(z_1 - C_1)$

$= -\frac{1}{\frac{4}{9}}\cdot\frac{7}{9} = -\frac{7}{4}$

$(C_3 + \Delta C_3)\left(-\frac{1}{9}\right) + C_2\cdot\frac{4}{3} + 0\cdot\left(-\frac{10}{9}\right) - 1$
$= \left(C_3\left(-\frac{1}{9}\right) + C_2\cdot\frac{4}{3} + 0\cdot\left(-\frac{10}{9}\right) - 1\right) - \Delta C_3\cdot\frac{1}{9} \ge 0$.
$= \frac{14}{9}$

$\Downarrow$

$\Delta C_3 \le \frac{\frac{14}{9}}{\frac{1}{9}} = -\frac{1}{t_{34}}(z_4 - C_4)$

$\max\left\{-\frac{7}{4}, -1\right\} \le \Delta C_3 \le \min\{14, 5\}$
$-1 \le \Delta C_3 \le 5$

$$\boxed{\max\left\{ -\frac{z_j - c_j}{t_{rj}} \,\middle|\, t_{rj} > 0 \right\} \le \Delta C_{i_r} \le \min\left\{ -\frac{z_j - c_j}{t_{rj}} \,\middle|\, t_{rj} < 0 \right\}}$$

$-\infty < \Delta C_1 \le \frac{7}{9}$
$-\frac{7}{6} \le \Delta C_2 < \infty$
$-1 \le \Delta C_3 \le 5$
$-\infty < \Delta C_4 \le \frac{14}{9}$