In[2]:=
```
Pivot[A_, k_, l_] := Module[{B = A},
m = Dimensions[A][[1]]; n = Dimensions[A][[2]];
For[i = 1, i ≤ m, i++,
If[i == k,
B[[i]] = A[[i]] / A[[k, l]],
  B[[i]] = -A[[i, l]] / A[[k, l]] * A[[k]] + A[[i]]]
];
B
]

(*3.2:Exercise 7*)
```

In[3]:=
```
MatrixForm[
 A1 = {{1, 1, 2, 1, 0, 0, 2}, {2, 3, 4, 0, 1, 0, 3}, {3, 3, 1, 0, 0, 1, 4}, {-8, -9, -5, 0, 0, 0, 0}}]
```

Out[3]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 2 \\ 2 & 3 & 4 & 0 & 1 & 0 & 3 \\ 3 & 3 & 1 & 0 & 0 & 1 & 4 \\ -8 & -9 & -5 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[4]:=
```
MatrixForm[A2 = Pivot[A1, 2, 2]]
```

Out[4]//MatrixForm=

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 1 & -\frac{1}{3} & 0 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} & 0 & \frac{1}{3} & 0 & 1 \\ 1 & 0 & -3 & 0 & -1 & 1 & 1 \\ -2 & 0 & 7 & 0 & 3 & 0 & 9 \end{pmatrix}$$

In[5]:=
```
MatrixForm[A3 = Pivot[A2, 3, 1]]
```

Out[5]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & \frac{5}{3} & 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{10}{3} & 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 1 & 0 & -3 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 11 \end{pmatrix}$$

In[10]:=
```
(*Dual Problem: Dual Simplex Method*)
MatrixForm[D1 = {{-1, -2, -3, 1, 0, 0, -8},
{-1, -3, -3, 0, 1, 0, -9},
{-2, -4, -1, 0, 0, 1, -5},
{2, 3, 4, 0, 0, 0, 0}}]
```

Out[10]//MatrixForm=

$$\begin{pmatrix} -1 & -2 & -3 & 1 & 0 & 0 & -8 \\ -1 & -3 & -3 & 0 & 1 & 0 & -9 \\ -2 & -4 & -1 & 0 & 0 & 1 & -5 \\ 2 & 3 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[7]:= `MatrixForm[D2 = Pivot[D1, 2, 2]]`

Out[7]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{3} & 0 & -1 & 1 & -\frac{2}{3} & 0 & -2 \\ \frac{1}{3} & 1 & 1 & 0 & -\frac{1}{3} & 0 & 3 \\ -\frac{2}{3} & 0 & 3 & 0 & -\frac{4}{3} & 1 & 7 \\ 1 & 0 & 1 & 0 & 1 & 0 & -9 \end{pmatrix}$$

In[8]:= `MatrixForm[D3 = Pivot[D2, 1, 3]]`

Out[8]//MatrixForm=

$$\begin{pmatrix} \frac{1}{3} & 0 & 1 & -1 & \frac{2}{3} & 0 & 2 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ -\frac{5}{3} & 0 & 0 & 3 & -\frac{10}{3} & 1 & 1 \\ \frac{2}{3} & 0 & 0 & 1 & \frac{1}{3} & 0 & -11 \end{pmatrix}$$

A	I	b

		$c_1 \;\cdots\; c_j \;\cdots$	
C_B	x_{i_1} ⋮ x_{i_m}	$t_1 \; t_2 \; - \; t_j \cdots \; t_n$	$x_B = B^{-1}b$
		$C_B^T t_1 - c_1 \cdots \; C_B^T t_j - c_j \cdots$	$C_B^T \cdot x_B$

$$\boxed{\text{Optimality} \Rightarrow \quad C_B^T \cdot t_j - c_j \geq 0}$$

Maximum $z = C_B^T \cdot x_B = C_B^T \cdot B^{-1} \cdot b = b^T (B^{-1})^T \cdot C_B = b^T \cdot w$

$w = (B^{-1})^T \cdot C_B$.

$w^T = C_B^T \cdot B^{-1}$.

		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	1	1	2	1	0	0	2
0	x_5	2	3	4	0	1	0	3
0	x_6	3	3	1	0	0	1	4
$z \rightarrow$		-8	-9	-5	0	0	0	0

		8	9	5	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	$\frac{1}{3}$	0	$\frac{2}{3}$	1	$-\frac{1}{3}$	0	1
9	x_2	$\frac{2}{3}$	1	$\frac{4}{3}$	0	$\frac{1}{3}$	0	1
0	x_6	1	0	-3	0	-1	1	1
		-2	0	7	0	3	0	9

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 3 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

basic feasible solution:

$(x_1, x_2, x_3, x_4, x_5, x_6)$

$\downarrow \; \downarrow \; \downarrow \; \downarrow \; \downarrow \; \downarrow$

$0, \; 1, \; 0, \; 1, \; 0, \; 1$

$$B x_B = b \Rightarrow x_B = B^{-1} b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$B^{-1} = \begin{pmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$z = C_B^T \cdot B^{-1} b = w^T \cdot b$$

$$= \begin{pmatrix} 0 \\ 9 \\ 0 \end{pmatrix}^T \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 9.$$

$$w = C_B^T \cdot B^{-1} = (0, 3, 0).$$

		8	9	5	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	0	$\frac{5}{3}$	1	0	$-\frac{1}{3}$	$\frac{2}{3}$
9	x_2	0	1	$\frac{10}{3}$	0	1	$-\frac{2}{3}$	$\frac{1}{3}$
8	x_1	1	0	-3	0	-1	1	1
		0	0	1	0	1	2	11

$$B=\begin{pmatrix}1&1&1\\0&3&2\\0&3&3\end{pmatrix},\quad B^{-1}=\begin{pmatrix}1&0&-\frac{1}{3}\\0&1&-\frac{2}{3}\\0&-1&1\end{pmatrix}$$

$$x_B=B^{-1}b=\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\\1\end{pmatrix}\begin{matrix}\leftarrow x_4\\\leftarrow x_2\\\leftarrow x_1\end{matrix}$$

$$z=C_B^T\cdot B^{-1}b$$
$$=\begin{pmatrix}0\\9\\8\end{pmatrix}^T\cdot\begin{pmatrix}\frac{2}{3}\\\frac{1}{3}\\1\end{pmatrix}=9\cdot\frac{1}{3}+8=11$$

$$z=x_B^T B^{-1}b=b^T\cdot\underbrace{(B^{-1})^T\cdot x_B}_{w}=b^T\cdot w$$

$w=(B^{-1})^T C_B$

$w^T=C_B^T\cdot B^{-1}$ is the optimal solution to the dual problem.

$\quad = (0,1,2)$.

<u>Dual Problem</u>: Minimize $z=2w_1+3w_2+4w_3$

subject to:
$$1\cdot w_1+2w_2+3w_3\geq 8$$
$$1\cdot w_1+3w_2+3w_3\geq 9$$
$$2\cdot w_1+4\cdot w_2+1\cdot w_3\geq 5$$
$$w_1,w_2,w_3\geq 0.$$

(w_1,w_2,w_3): $(0,0,0)\longrightarrow(0,3,0)\longrightarrow(0,1,2)$

$(w_1,\dots,w_6)$: $(0,0,0,-8,-9,-5)\longrightarrow(0,3,0,-2,0,7)\longrightarrow(0,1,2,0,0,7)$

__Dual Problem__: Minimize $z = 2w_1 + 3w_2 + 4w_3$

subject to: $1\cdot w_1 + 2w_2 + 3w_3 \geq 8 \rightsquigarrow -w_1 - 2w_2 - 3w_3 \leq -8$

$1\cdot w_1 + 3w_2 + 3w_3 \geq 9 \rightsquigarrow -w_1 - 3w_2 - 3w_3 \leq -9$

$2\cdot w_1 + 4\cdot w_2 + 1\cdot w_3 \geq 5 \rightsquigarrow -2w_1 - 4w_2 - 1\cdot w_3 \leq -5$

$w_1, w_2, w_3 \geq 0.$

$\rightsquigarrow$ Maximize $z' = -2w_1 - 3w_2 - 4w_3$

Subject to $-w_1 - 2w_2 - 3w_3 + w_4 = -8$

$-w_1 - 3w_2 - 3w_3 + w_5 = -9$

__Dual Simplex Method__

$-2w_1 - 4\cdot w_2 - 1\cdot w_3 + w_6 = -5$

	w_1	w_2 ↓	w_3	w_4	w_5	w_6	
w_4	-1	-2	-3	$+1$	0	0	-8
← w_5	-1	-3	-3	0	$+1$	0	-9
w_6	-2	-4	-1	0	0	$+1$	-5
	2	3	4	0	0	0	0
	↑	↑	↑				
	-2	-1	$-\frac{4}{3}$				

Minimize

$z = b_1w_1 + b_2w_2 + \cdots + b_mw_m$

$b_i \geq 0.$

	<0
	≥ 0
$b_1 \quad b_2 \quad b_3 \cdots b_m \quad 0 \cdots$	$0.$

LP $\underset{m\times n}{A}x \leq b.$ $x \geq 0.$ $\rightsquigarrow$ $Ax \leq b$

$a_{m+1,1}x_1 + \cdots + a_{m+1,n}x_n \leq b_{m+1}$

	≥ 0
	≥ 0
$a_{m+1,1} \cdots a_{m+1,n}$	<0
$\geq 0 \;\; \geq 0$	

Maximize: $z = 8x_1 + 9x_2 + 5x_3$

subject to:

$$x_1 + x_2 + 2x_3 \le 2 \rightsquigarrow x_1 + x_2 + 2x_3 + x_4 = 2$$
$$2x_1 + 3x_2 + 4x_3 \le 3 \rightsquigarrow 2x_1 + 3x_2 + 4x_3 + x_5 = 3$$
$$3x_1 + 3x_2 + x_3 \le 4 \rightsquigarrow 3x_1 + 3x_2 + x_3 + x_6 = 4$$
$$x_j \ge 0,\ j = 1, 2, 3. \qquad x_i \ge 0,\ i = 1, \cdots, 6.$$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	2	1	0	0	2
x_5	2	3	4	0	1	0	3
x_6	3	3	1	0	0	1	4
	-8	-9	-5	0	0	0	0

		8	9	5	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	0	$\frac{5}{3}$	1	0	$-\frac{1}{3}$	$\frac{2}{3}$
9	x_2	0	1	$\frac{10}{3}$	0	1	$-\frac{2}{3}$	$\frac{1}{3}$
8	x_1	1	0	-3	0	-1	1	1
		0	0	1	0	1	2	11

$$(x_1, x_2, x_3, x_4, x_5, x_6) = (1, \tfrac{1}{3}, 0, \tfrac{2}{3}, 0, 0) \qquad (w_1, w_2, w_3) = (0, 1, 2)$$

$w_4 = 0 = w_5$ $\quad$ $0 \ne x_4 \Rightarrow w_1 = 0.$

Dual Problem: Minimize $z = 2w_1 + 3w_2 + 4w_3$

subject to:

$$1 \cdot w_1 + 2w_2 + 3w_3 \ge 8 \rightsquigarrow -w_1 - 2w_2 - 3w_3 \le -8$$
$$1 \cdot w_1 + 3w_2 + 3w_3 \ge 9 \rightsquigarrow -w_1 - 3w_2 - 3w_3 \le -9$$
$$2 \cdot w_1 + 4 \cdot w_2 + 1 \cdot w_3 \ge 5 \rightsquigarrow -2w_1 - 4w_2 - 1 \cdot w_3 \le -5$$
$$w_1, w_2, w_3 \ge 0.$$

Thm: For any pair of optimal solutions to the primal and dual Problem product of the i-th slack variable for the primal problem and the i-th dual variable $= 0$.