In[27]:=

```
Pivot[A_, k_, l_] := Module[{B = A},
m = Dimensions[A][[1]]; n = Dimensions[A][[2]];
For[i = 1, i ≤ m, i++,
If[i == k,
B[[i]] = A[[i]] / A[[k, l]],
  B[[i]] = -A[[i, l]] / A[[k, l]] * A[[k]] + A[[i]]]
];
B
]
```

```
(*3.2: Exercise 7*)
```

In[29]:=

```
MatrixForm[
 A1 = {{1, 1, 2, 1, 0, 0, 2}, {2, 3, 4, 0, 1, 0, 3}, {3, 3, 1, 0, 0, 1, 4}, {-8, -9, -5, 0, 0, 0, 0}}]
```

Out[29]//MatrixForm=

$$\begin{pmatrix} 1 & 1 & 2 & 1 & 0 & 0 & 2 \\ 2 & 3 & 4 & 0 & 1 & 0 & 3 \\ 3 & 3 & 1 & 0 & 0 & 1 & 4 \\ -8 & -9 & -5 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[30]:=

```
MatrixForm[A2 = Pivot[A1, 2, 2]]
```

Out[30]//MatrixForm=

$$\begin{pmatrix} \frac{1}{3} & 0 & \frac{2}{3} & 1 & -\frac{1}{3} & 0 & 1 \\ \frac{2}{3} & 1 & \frac{4}{3} & 0 & \frac{1}{3} & 0 & 1 \\ 1 & 0 & -3 & 0 & -1 & 1 & 1 \\ -2 & 0 & 7 & 0 & 3 & 0 & 9 \end{pmatrix}$$

In[31]:=

```
MatrixForm[A3 = Pivot[A2, 3, 1]]
```

Out[31]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & \frac{5}{3} & 1 & 0 & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{10}{3} & 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ 1 & 0 & -3 & 0 & -1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 2 & 11 \end{pmatrix}$$

In[34]:=

```
B = {{1, 1, 1}, {0, 3, 2}, {0, 3, 3}};
Inverse[B].{2, 4, 1}
```

Out[35]=

$$\left\{\frac{5}{3}, \frac{10}{3}, -3\right\}$$

In[36]:=
```
(*Dual problem*)
MatrixForm[B0 = {{1, 2, 3, -1, 0, 0, 1, 0, 0, 8}, {1, 3, 3, 0, -1, 0, 0, 1, 0, 9},
   {2, 4, 1, 0, 0, -1, 0, 0, 1, 5}, {0, 0, 0, 0, 0, 0, 1, 1, 1, 0}}]
```

Out[36]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 & -1 & 0 & 0 & 1 & 0 & 0 & 8 \\ 1 & 3 & 3 & 0 & -1 & 0 & 0 & 1 & 0 & 9 \\ 2 & 4 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

In[37]:=
```
B1 = B0; B1[[4]] = B0[[4]] - B0[[1]] - B0[[2]] - B0[[3]];
MatrixForm[B1]
```

Out[38]//MatrixForm=

$$\begin{pmatrix} 1 & 2 & 3 & -1 & 0 & 0 & 1 & 0 & 0 & 8 \\ 1 & 3 & 3 & 0 & -1 & 0 & 0 & 1 & 0 & 9 \\ 2 & 4 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 5 \\ -4 & -9 & -7 & 1 & 1 & 1 & 0 & 0 & 0 & -22 \end{pmatrix}$$

In[39]:=
```
MatrixForm[B2 = Pivot[B1, 3, 2]]
```

Out[39]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & \frac{5}{2} & -1 & 0 & \frac{1}{2} & 1 & 0 & -\frac{1}{2} & \frac{11}{2} \\ -\frac{1}{2} & 0 & \frac{9}{4} & 0 & -1 & \frac{3}{4} & 0 & 1 & -\frac{3}{4} & \frac{21}{4} \\ \frac{1}{2} & 1 & \frac{1}{4} & 0 & 0 & -\frac{1}{4} & 0 & 0 & \frac{1}{4} & \frac{5}{4} \\ \frac{1}{2} & 0 & -\frac{19}{4} & 1 & 1 & -\frac{5}{4} & 0 & 0 & \frac{9}{4} & -\frac{43}{4} \end{pmatrix}$$

In[41]:=
```
MatrixForm[B3 = Pivot[B2, 1, 3]]
```

Out[41]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5} & \frac{2}{5} & 0 & -\frac{1}{5} & \frac{11}{5} \\ -\frac{1}{2} & 0 & 0 & \frac{9}{10} & -1 & \frac{3}{10} & -\frac{9}{10} & 1 & -\frac{3}{10} & \frac{3}{10} \\ \frac{1}{2} & 1 & 0 & \frac{1}{10} & 0 & -\frac{3}{10} & -\frac{1}{10} & 0 & \frac{3}{10} & \frac{7}{10} \\ \frac{1}{2} & 0 & 0 & -\frac{9}{10} & 1 & -\frac{3}{10} & \frac{19}{10} & 0 & \frac{13}{10} & -\frac{3}{10} \end{pmatrix}$$

In[42]:=
```
MatrixForm[B4 = Pivot[B3, 2, 4]]
```

Out[42]//MatrixForm=

$$\begin{pmatrix} -\frac{2}{9} & 0 & 1 & 0 & -\frac{4}{9} & \frac{1}{3} & 0 & \frac{4}{9} & -\frac{1}{3} & \frac{7}{3} \\ -\frac{5}{9} & 0 & 0 & 1 & -\frac{10}{9} & \frac{1}{3} & -1 & \frac{10}{9} & -\frac{1}{3} & \frac{1}{3} \\ \frac{5}{9} & 1 & 0 & 0 & \frac{1}{9} & -\frac{1}{3} & 0 & -\frac{1}{9} & \frac{1}{3} & \frac{2}{3} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

In[43]:=
```
C0 = Drop[B4, None, {7, 9}]; C0[[4]] = {2, 3, 4, 0, 0, 0, 0};
MatrixForm[C0]
```

Out[44]//MatrixForm=

$$\begin{pmatrix} -\frac{2}{9} & 0 & 1 & 0 & -\frac{4}{9} & \frac{1}{3} & \frac{7}{3} \\ -\frac{5}{9} & 0 & 0 & 1 & -\frac{10}{9} & \frac{1}{3} & \frac{1}{3} \\ \frac{5}{9} & 1 & 0 & 0 & \frac{1}{9} & -\frac{1}{3} & \frac{2}{3} \\ 2 & 3 & 4 & 0 & 0 & 0 & 0 \end{pmatrix}$$

In[45]:=
```
C1 = C0; C1[[4]] = C0[[4]] - 4 * C0[[1]] - 3 * C0[[3]];
MatrixForm[C1]
```

Out[46]//MatrixForm=

$$\begin{pmatrix} -\frac{2}{9} & 0 & 1 & 0 & -\frac{4}{9} & \frac{1}{3} & \frac{7}{3} \\ -\frac{5}{9} & 0 & 0 & 1 & -\frac{10}{9} & \frac{1}{3} & \frac{1}{3} \\ \frac{5}{9} & 1 & 0 & 0 & \frac{1}{9} & -\frac{1}{3} & \frac{2}{3} \\ \frac{11}{9} & 0 & 0 & 0 & \frac{13}{9} & -\frac{1}{3} & -\frac{34}{3} \end{pmatrix}$$

In[47]:=
```
MatrixForm[C2 = Pivot[C1, 2, 6]]
```

Out[47]//MatrixForm=

$$\begin{pmatrix} \frac{1}{3} & 0 & 1 & -1 & \frac{2}{3} & 0 & 2 \\ -\frac{5}{3} & 0 & 0 & 3 & -\frac{10}{3} & 1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 \\ \frac{2}{3} & 0 & 0 & 1 & \frac{1}{3} & 0 & -11 \end{pmatrix}$$

3.2: Exercise 7:

Maximize: $z = 8x_1 + 9x_2 + 5x_3$

subject to: $x_1 + x_2 + 2x_3 \le 2 \rightsquigarrow x_1 + x_2 + 2x_3 + x_4 = 2$

$2x_1 + 3x_2 + 4x_3 \le 3 \rightsquigarrow 2x_1 + 3x_2 + 4x_3 + x_5 = 3$

$3x_1 + 3x_2 + x_3 \le 4 \rightsquigarrow 3x_1 + 3x_2 + x_3 + x_6 = 4$

$x_j \ge 0,\ j = 1, 2, 3.$ $\quad x_i \ge 0,\ i = 1, \dots, 6.$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	2	1	0	0	2
x_5	2	3	4	0	1	0	3
x_6	3	3	1	0	0	1	4
	-8	-9	-5	0	0	0	0

		8	9	5	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	0	$\frac{5}{3}$	1	0	$-\frac{1}{3}$	$\frac{2}{3}$
9	x_2	0	1	$\frac{10}{3}$	0	1	$-\frac{2}{3}$	$\frac{1}{3}$
8	x_1	1	0	-3	0	-1	1	1
		0	0	1	0	1	2	11

$\rightsquigarrow$ Optimal solution $(x_1, x_2, x_3, x_4, x_5, x_6) = (1, \frac{1}{3}, 0, \frac{2}{3}, 0, 0)$

Maximum $z = 11$.

basic feasible optimal.

basic variables:

$(x_4, x_2, x_1) \rightsquigarrow B = \begin{array}{c} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & 3 & 3 \end{pmatrix} \\ \begin{matrix} x_4 & x_2 & x_1 \end{matrix} \end{array}$, $\quad B x_B = b = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$.

$\Downarrow$

$x_B = B^{-1} \cdot b$.

$B^{-1} = \begin{pmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & -1 & 1 \end{pmatrix}$

$c_B^T \cdot B^{-1} = (0, 9, 8) \cdot B^{-1} = (0, 1, 2) = w^T$

Maximize: $z = 8x_1 + 9x_2 + 5x_3$

subject to: $1 x_1 + 1 \cdot x_2 + 2x_3 \le 2$

$2x_1 + 3x_2 + 4x_3 \le 3$

$3x_1 + 3x_2 + x_3 \le 4$

$x_j \ge 0,\ j = 1, 2, 3.$

$z = C^T \cdot x$

$Ax \le b$

$x \ge 0$.

Dual Problem: Minimize $z = 2w_1 + 3w_2 + 4w_3$

subject to: $1 \cdot w_1 + 2w_2 + 3w_3 \geq 8$

$1 \cdot w_1 + 3w_2 + 3w_3 \geq 9$

$2 \cdot w_1 + 4 \cdot w_2 + 1 \cdot w_3 \geq 5$

$w_1, w_2, w_3 \geq 0$.

$b^T \cdot w$

$A^T \cdot w \geq c$.

$w \geq 0$.

$\rightsquigarrow$ Maximize $z' = -2w_1 - 3w_2 - 4w_3$

Subject to $w_1 + 2w_2 + 3w_3 - w_4 + v_1 = 8$

$w_1 + 3w_2 + 3w_3 - w_5 + v_2 = 9$

$2w_1 + 4 \cdot w_2 + 1 \cdot w_3 - w_6 + v_3 = 5$

$w_i \geq 0, \; v_j \geq 0$.

Phase 1: Minimize $v_1 + v_2 + v_3$

$\Updownarrow$

Maximize $-v_1 - v_2 - v_3$

		0	0	0	0	0	0	-1	-1	-1	
C_B	X_B	w_1	w_2	w_3	w_4	w_5	w_6	v_1	v_2	v_3	
-1	v_1	1	2	3	-1	0	0	1	0	0	8
-1	v_2	1	3	3	0	-1	0	0	1	0	9
-1	v_3	2	4	1	0	0	-1	0	0	1	5
		-4	-9	-7	1	1	1	0	0	0	-22
		~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~0~~	~~1~~	~~1~~	~~1~~	

Final Tableau Phase 2:

$\rightsquigarrow$ Optimal solution $(w_1, w_2, w_3, w_4, w_5, w_6) = (0, 1, 2, 0, 0, 1)$

Maximum of $z' = -11$

$\Updownarrow$

Minimum of $z = 2w_1 + 3w_2 + 4w_3 = 11$

Maximize: $z = 8x_1 + 9x_2 + 5x_3$

subject to: $x_1 + x_2 + 2x_3 \le 2 \rightsquigarrow x_1 + x_2 + 2x_3 + x_4 = 2$

$2x_1 + 3x_2 + 4x_3 \le 3 \rightsquigarrow 2x_1 + 3x_2 + 4x_3 + x_5 = 3$

$3x_1 + 3x_2 + x_3 \le 4 \rightsquigarrow 3x_1 + 3x_2 + x_3 + x_6 = 4$

$x_j \ge 0,\ j = 1, 2, 3.$ $\quad x_i \ge 0,\ i = 1, \cdots, 6.$

	x_1	x_2	x_3	x_4	x_5	x_6	
x_4	1	1	2	1	0	0	2
x_5	2	3	4	0	1	0	3
x_6	3	3	1	0	0	1	4
	-8	-9	-5	0	0	0	0

		8	9	5	0	0	0	
		x_1	x_2	x_3	x_4	x_5	x_6	
0	x_4	0	0	$\frac{5}{3}$	1	0	$-\frac{1}{3}$	$\frac{2}{3}$
9	x_2	0	1	$\frac{10}{3}$	0	1	$-\frac{2}{3}$	$\frac{1}{3}$
8	x_1	1	0	-3	0	-1	1	1
		0	0	1	0	1	2	11

$11 = C_B^T x_B = C_B^T B^{-1} b$, $\quad C_B^T x_B = (0\ 9\ 8)\cdot\begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 1 \end{pmatrix}$

Optimal solution $(w_1, w_2, w_3, w_4, w_5, w_6) = (0, 1, 2, 0, 0, 1)$

Complementary slackness

(i-th slack) x (i-th dual variable) = 0

Maximum of $z' = -11$

$\Updownarrow$

Minimum of $z = 2w_1 + 3w_2 + 4w_3 = 11 = b^T \cdot w$

$b^T \cdot w = (2\ 3\ 4)\cdot\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = b^T \cdot (B^{-1})^T C_B$

Dual Problem: Minimize $z = 2w_1 + 3w_2 + 4w_3$

subject to: $1\cdot w_1 + 2w_2 + 3w_3 \ge 8$

$1\cdot w_1 + 3w_2 + 3w_3 \ge 9$

$2\cdot w_1 + 4\cdot w_2 + 1\cdot w_3 \ge 5$

$w_1, w_2, w_3 \ge 0.$

Thm (weak Duality Thm):

Maximize $z = C^T x$

subject to $Ax \le b.$

$x \ge 0$

$\uparrow$

x_0 feasible

Minimize $z' = b^T w$

$A^T w \ge C$

$w \ge 0.$

$\uparrow$

w_0 feasible

Then: $C^T x_0 \le b^T \cdot w_0$

Pf:

Assume x_0 feasible $\Rightarrow$ $A\cdot x_0 \le b$
$0 \le w_0$ feasible

$$\Rightarrow w_0^T \cdot A x_0 \le w_0^T \cdot b$$

$$w_0^T A x_0 = x_0^T A^T \cdot w_0 \ge x_0^T \cdot C = C^T x_0 \quad (x_0 \ge 0)$$

$$w_0^T \cdot b \ge C^T x_0$$

Cor: x_0 feasible, w_0 feasible, if $C^T x_0 = b^T \cdot w_0$, then x_0 and w_0 are optimal.

Thm (Duality Thm): (a) If either the primal problem or dual problem has a feasible solution with a finite optimal value, then the other problem has a feasible solution with the same optimal value.

(b) If primal and dual have feasible solutions, then both of them have optimal solutions with the same optimal value.

Pf of (a): Maximize $z = C^T x = (C^T | 0)\binom{x}{x'} = z$

subject to $Ax \le b$. $\rightsquigarrow$ $Ax + x' = b = (A|I)\binom{x}{x'}$

$x \ge 0$ $\qquad \binom{x}{x'} \ge 0$.

If $\hat{x}$ is optimal solution, $\rightsquigarrow$ $x_B = B^{-1} \cdot b$. j-th column of the final tableaux

$\ni (x_{i_1}, \dots, x_{i_m}) \rightsquigarrow B$ $\qquad B^{-1}(A|I)_j = t_j$.

$$\begin{array}{c|c:c|c} & A & I & b \\ \hline & & & \end{array}$$

$$\begin{array}{cc|cccccc|c} & & c_1 & \cdots & c_j & \cdots & & & \\ \hline c_B & \begin{matrix} x_{r_1} \\ \vdots \\ x_{r_m} \end{matrix} & t_1 & t_2 & \cdots & t_j & \cdots & t_n & x_B = B^{-1}b \\ \hline & & c_B^T t_1 - c_1 & \cdots & c_B^T t_j - c_j & \cdots & \cdots & & c_B^T \cdot x_B \end{array}$$

$$\boxed{\text{Optimality} \Rightarrow \quad c_B^T \cdot t_j - c_j \geq 0}$$

$$\text{Maximum} \quad z = c_B^T \cdot x_B = c_B^T \cdot B^{-1} \cdot b = b^T (B^{-1})^T \cdot c_B = b^T \cdot w$$

$$w = (B^{-1})^T \cdot c_B .$$

$$w^T = c_B^T \cdot B^{-1} .$$