Dual Problem

Primal LP Problem in standard form:

$$c=\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}, \quad x=\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Maximize $\quad z = c_1 \cdot x_1 + c_2 x_2 + \cdots + c_n \cdot x_n = \boxed{c^T \cdot x}$

subject to :

m constr. $\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2 \\ \vdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_m \end{cases} \Leftrightarrow \boxed{A \cdot x \le b}$

$m \times n \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & \cdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$

$$\boxed{x_i \ge 0, \quad i = 1, \cdots, n.}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{pmatrix}$$

Dual Problem: Minimize $\quad b^T \cdot w$

$$\| $$

$$z' = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$$

subject to: $\underset{n \times m}{A^T} \cdot \underset{m \times 1}{w} \ge c, \quad w \ge 0.$

standard form

dual

Maximize $z = -b^T w$

subject to $(-A^T) w \leq -c, \quad w \geq 0.$

dual

dual of dual

Minimize $z' = (-c)^T \cdot x = -c^T \cdot x$

subject to $\underbrace{(-A^T)^T \cdot x}_{= -A \cdot x} \geq -b, \quad x \geq 0.$

$\Updownarrow$

$$\boxed{\begin{array}{ll} \text{Maximize} & z = c^T \cdot x \\ \text{subject to} & A \cdot x \leq b, \quad x \geq 0. \end{array}}$$

Primal Problem $\Longleftrightarrow$ dual Problem
$\parallel$
dual of dual.

Compare: $T: V \to W \rightsquigarrow T^*: W^* \to V^*$

$(T^*)^*: (V^*)^* \to (W^*)^*$

Ex: Maximize $z = 2x_1 + 3x_2$

subject to
$$3x_1 + 1 \cdot x_2 \le 6$$
$$1x_1 + 1 \cdot x_2 \le 4$$
$$1x_1 + 2x_2 \le 7$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \quad 3\times 2$$

$$A^T = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad 2\times 3.$$

Dual Problem: Minimize: $6w_1 + 4w_2 + 7w_3$

subject to:
$$3w_1 + 1 \cdot w_2 + 1 \cdot w_3 \ge 2$$
$$1 \cdot w_1 + 1 \cdot w_2 + 2 \cdot w_3 \ge 3.$$
$$w_1, w_2, w_3 \ge 0.$$

(# variables $\updownarrow$ # constraints)

Ex: Maximize $z = 2x_1 + 3x_2$

subject to
$$3x_1 + 1 \cdot x_2 = 6$$
$$1x_1 + 1 \cdot x_2 \le 4$$
$$1x_1 + 2x_2 \le 7$$
$$x_1 \ge 0,\ x_2 \ge 0.$$

$\Longleftrightarrow$

$$\begin{cases} 3x_1 + 1 \cdot x_2 \le 6 \\ -3x_1 - 1\, x_2 \le -6 \\ 1x_1 + x_2 \le 4 \\ 1x_1 + 2x_2 \le 7. \end{cases}$$
$$x_1, x_2 \ge 0.$$

Dual Problem: Minimize
$$z' = 6u_1 - 6u_2 + 4u_3 + 7u_4 = 6(\underbrace{u_1 - u_2}_{w_1}) + 4\underbrace{u_3}_{w_2} + 7\underbrace{u_4}_{w_3}$$

Subject to
$$3u_1 - 3u_2 + 1 \cdot u_3 + 1 \cdot u_4 \ge 2 \qquad 3w_1 + w_2 + w_3 \ge 2$$
$$1 \cdot u_1 - 1u_2 + 1 \cdot u_3 + 2u_4 \ge 3 \qquad w_1 + w_2 + 2w_3 \ge 3$$
$$\underbrace{u_1, u_2}_{w_1 \text{ unrestrictive}},\ \underbrace{u_3}_{w_2},\ \underbrace{u_4}_{w_3} \ge 0.$$

$$w_1 = \underset{\substack{\vee\!/ \\ 0}}{u_1} - \underset{\substack{\vee\!/ \\ 0}}{u_2}$$

Dual Problem:
$$z' = 6w_1 + 4w_2 + 7w_3$$
$$3w_1 + w_2 + w_3 \ge 2$$
$$w_1 + w_2 + 2w_3 \ge 3.$$

w_1 unrestrictive

$w_2, w_3 \ge 0.$

Ex: Maximize $z = 3x_1 + 2x_2 + x_3$
subject to $x_1 + 2x_2 - x_3 \leq 4$
$2x_1 - x_2 + x_3 = 8$
$x_1 \geq 0,\ x_2 \geq 0,\ x_3$ unrestrictive

Dual Problem: Minimize $z' = 4w_1 + 8w_2$
subject to: $1 \cdot w_1 + 2 \cdot w_2 \geq 3$ ($\leftrightarrow x_1$)
$2 \cdot w_1 - 1 \cdot w_2 \geq 2$. ($\leftrightarrow x_2$)
$-1 \cdot w_1 + 1 \cdot w_2 = 1$ ($\leftrightarrow x_3$)
$\leftrightarrow x_3$ is unrestrictive
$w_1 \geq 0,\ w_2$ unrestrictive.

Maximize $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$ x_j : # of j-th product. c_j profit of j-th output.

subject to
$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \leq b_1$ ← 1st. resource.
$\cdots$
$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \leq b_m$ m-th resource

a_{ij}: the amount of i-th resource needed to produce one unit of j-th product

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Minimze $z' = b_1 w_1 + b_2 w_2 + \cdots + b_m w_m$
subject to $a_{11} w_1 + a_{21} w_2 + \cdots + a_{m1} w_m \geq c_1$
$\cdots$
$a_{1m} w_1 + a_{2n} w_2 + \cdots + a_{mn} W_m \geq c_m$.
$w_j \geq 0$.

(w_i: fictitious prices
shadow prices.
$\cdots$)