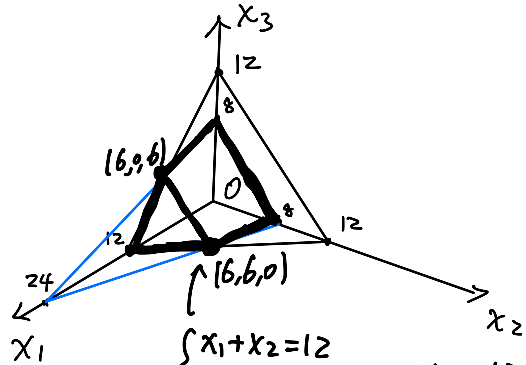
Maximize $z = 2x_1 + 4x_2 + 3x_3$

Subject to $\underline{x_1 + x_2 + x_3 \leq 12}$

$\underline{x_1 + 3x_2 + 3x_3 \leq 24}$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$



$\begin{cases} x_1 + x_2 = 12 \\ x_1 + 3x_2 = 24 \end{cases} \Rightarrow 2x_2 = 12 \Downarrow x_2 = 6,\ x_1 = 6$

Extremal points:

$(12,0,0),\ \underline{(6,6,0),\ (6,0,6)},\ (0,8,0),\ (0,0,8),\ (0,0,0)$

$z = 24,\ 36,\ 36,\ 32,\ 32,\ 0$

$(6r+6(1-r),\ 6r+0,\ 0+6(1-r))$

Optimal solutions: $(6,6,0),\ (6,0,6).$

$\underline{r\cdot(6,6,0) + (1-r)(6,0,6)} = (6,\ 6r,\ 6(1-r))$

$0 \leq r \leq 1.$

Maximum: 36

$\Rightarrow$ Canonical Form:

$x_1 + x_2 + x_3 + x_4 = 12$

$x_1 + 3x_2 + 3x_3 + x_5 = 24$

$x_i \geq 0,\ i = 1, \ldots, 5.$

$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 3 & 0 & 1 \end{pmatrix}$

$\uparrow\ \uparrow$

		x_1	x_2	x_3	x_4	x_5
$(12,0,0)$	$\longrightarrow$	$(12,$	$0,$	$0,$	$0,$	$12)$
$(6,6,0)$	$\longrightarrow$	$(6,$	$6,$	$0,$	$0,$	$0)$
$(6,0,6)$	$\longrightarrow$	$(6,$	$0,$	$6,$	$0,$	$0)$
$(0,0,8)$	$\longrightarrow$	$(0,$	$0,$	$8,$	$4,$	$0)$
$(0,0,0)$	$\longrightarrow$	$(0,$	$0,$	$0,$	$12,$	$24)$

Def: (basic solution to $Ax = b$)

$A = (A_1\ A_2 \cdots A_n)$ $m \times n$ matrix.

$\underline{A_{i_1}, \cdots, A_{i_m}}$ linearly independent $\Rightarrow$ $\underbrace{B = (A_{i_1} \cdots A_{i_m})}_{m \times m}$

$$\begin{pmatrix} x_{Bi_1} \\ x_{Bi_2} \\ \vdots \\ x_{Bi_m} \end{pmatrix} = x_B = B^{-1}b \quad (\Leftarrow B \cdot x_B = b).$$

$$\begin{array}{ccccccccccccc} x_1 & \cdots & x_{i_1-1} & x_{i_1} & x_{i_1+1} & \cdots & \cdots & x_{i_2} & & \cdots & & x_{i_m} & \cdots \end{array}$$

$\rightsquigarrow (0, \cdots, 0, x_{Bi_1}, 0 \cdots, 0, x_{Bi_2}, \cdots, x_{Bi_m}, 0, \cdots, 0)$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 3 & 3 & 0 & 1 \end{pmatrix} \qquad B = (A_1\ A_3) = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}.$$

$\uparrow$ A_1 $\uparrow$ A_3

$$B \cdot x = b = \begin{pmatrix} 12 \\ 24 \end{pmatrix} \Rightarrow x_B = \begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 12 \\ 24 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix} \cdot x \qquad = \frac{1}{2} \cdot \begin{pmatrix} 3 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 12 \\ 24 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$

$$\begin{array}{ccccc} x_1 & x_2 & x_3 & x_4 & x_5 \end{array}$$

$$\Rightarrow \underline{x = (6, 0, 6, 0, 0)} \Leftrightarrow (A_1\ A_3)$$

feasible $x_i \geq 0$. basic feasible solution.

$Bx = b$ $\quad x_i \geq 0$.

basic feasible solution. $\Leftarrow Ax = b$.

Ex: $A = \begin{pmatrix} 2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 6 & 1 & 0 & 3 \end{pmatrix}$, 3×5 $\qquad b = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$.

$x = (1, 0, -1, 1, 0)$.

$$Ax = 1\cdot\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} - 1\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2-1+0 \\ -1-0+2 \\ 0-1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \neq b.$$

$x = (0, 2, -5, 0, -1)$

$$A\cdot x = 2\cdot\begin{pmatrix} 3 \\ 1 \\ 6 \end{pmatrix} - 5\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 1\cdot\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 6-5-0 \\ 2-0-1 \\ 12-5-3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix} = b.$$

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 6 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

Not a basic solution.

$$\begin{array}{c} \quad x_1 \;\; x_2 \;\; x_3 \;\; x_4 \;\; x_5 \\ x = (0, 0, 1, 0, 1) \end{array} \qquad Ax = 1\cdot\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1\cdot\begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$$

3 cases: (1) x_1, x_2: nonbasic variable.

x_3, x_4, x_5: basic variables.

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \text{ invertible}$$

x is a basic solution if x_3, x_4, x_5 are basic variables.

2. x_1, x_4 : nonbasic variable

$\underline{x_2}, \underline{x_3}, \underline{x_5}$: basic variables.

$$A = \begin{pmatrix} 2 & 3 & 1 & 0 & 0 \\ -1 & 1 & 0 & 2 & 1 \\ 0 & 6 & 1 & 0 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 0 & 1 \\ 6 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & -3 \\ 1 & 0 & 1 \\ 0 & 1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{pmatrix} \quad rk = 2 < 3$$

Not a basic solution if x_2, x_3, x_5 are basic var.

3. x_2, x_4 : nonbasic var.

$$x = (\underset{\substack{\uparrow}}{0}, \underset{\substack{\uparrow \\ \text{nonbasic}}}{0}, \overset{\text{basic}}{1}, 0, 1)$$

$\underline{x_1}, \underline{x_3}, \underline{x_5}$: basic var.

$$B = \begin{pmatrix} 2 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{invertible}$$

$\Rightarrow$ x is a basic solution if x_1, x_3, x_5 are basic variables.

<u>Thm</u>: LPP in canonical form:

$$D = \{x \in \mathbb{R}^n : Ax = b, \ x \geq 0\} = \text{set of feasible solution.}$$

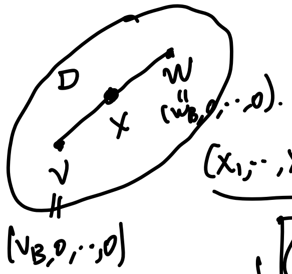
A point $x = (x_1, \cdots, x_n) \in D$ is an extremal point <u>if and only if</u> it is a basic feasible solution.

<u>Pf</u>: <u>basic</u> feasible solution $\Rightarrow$ extremal point.

$x = (x_1, \cdots, x_m, 0, \cdots, 0)$ solves $Bx = b$, $B = (A_1 A_2 \cdots A_m)$.

$x_i \geq 0$. $i = 1, \cdots, m$.

<u>WLOG</u>



$$\underline{x = (1-t)v + t \cdot w} . \quad 0 < t < 1$$

$$\underbrace{(x_1, \cdots, x_m, 0, \cdots, 0)} = (1-t)(v_1, \cdots, v_m, v_{m+1}, \cdots, v_n) + t \cdot (w_1, \cdots, w_m, w_{m+1}, \cdots, w_n)$$

$$\begin{cases} \boxed{(x_1, \cdots, x_m) = (1-t)(v_1, \cdots, v_m) + t \cdot (w_1, \cdots, w_m)} \\ \underbrace{(0, \cdots, 0)} = \underbrace{(1-t)}_{>0} \cdot \underbrace{(v_{m+1}, \cdots, v_n)}_{\geq 0} + \underbrace{t}_{>0} \cdot \underbrace{(w_{m+1}, \cdots, w_n)}_{\geq 0} \end{cases}$$

$$b = B \cdot x_B = B \cdot v_B = B \cdot w_B \Rightarrow v_B = x_B = w_B .$$

$$\Downarrow$$

invertible. $\Rightarrow x_B = B^{-1} b$.

$$v = x = w .$$

Extremal point of D $\Longrightarrow$ basic feasible solution.

$$\underline{x = (\underset{\substack{\vee \\ 0 \\ \downarrow \\ A_1}}{x_1}, \cdots, \underset{\substack{\vee \\ 0 \\ \downarrow \\ A_k}}{x_k}, 0, \cdots, 0)} \Longrightarrow A_1, \cdots, A_k \text{ linearly independent.}$$

Claim

$$\boxed{(A_1 \cdots A_k\ A_{k+1} \cdots A_m) \begin{pmatrix} x_1 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix} = b}$$

If not linearly independent. $c_1 A_1 + c_2 A_2 + \cdots + c_k A_k = 0$.

$c_1 \neq 0$.

$$\underline{x_1 A_1 + x_2 A_2 + \cdots + x_k A_k = b}$$

$$(\underbrace{x_1 + c_1 \cdot \varepsilon}_{>0}) A_1 + (\underbrace{x_2 + c_2 \cdot \varepsilon}_{>0}) A_2 + \cdots + (\underbrace{x_k + c_2 \cdot \varepsilon}_{>0}) A_k = b .$$

$|\varepsilon| \ll 1$.