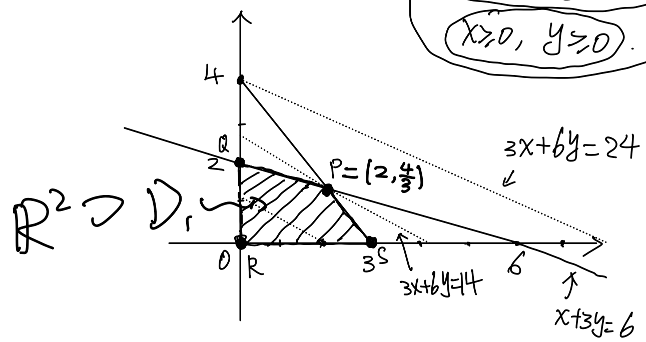
Ex: Maximize $z = 3x + 6y$

Subject to $x + 3y \le 6$ $\leadsto$ $x+3y=6$

$4x + 3y \le 12$ $\to$ $4x+3y=12$

$x \ge 0,\ y \ge 0$.



$3x+6y=6$

$x+2y=2$

Extremal Point Thm: If there is an optimal solution, then there must be an optimal solution that occurs at an extremal point of the set of feasible solution.

- transform into the canonical form:

$$\left.\begin{aligned} &x + 3y + u = 6 \\ &4x + 3y + v = 12 \\ &x, y \ge 0,\ u_1 \ge 0,\ u_2 \ge 0. \end{aligned}\right\} \leadsto D_2 \subset \mathbb{R}^4$$

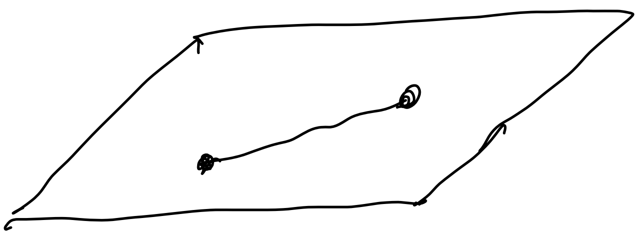
convex.

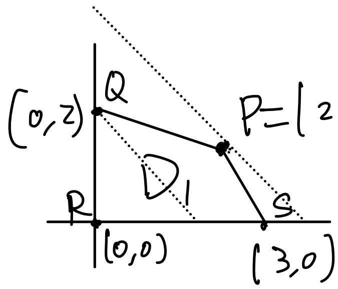
$$(x, y) \in D_1 \iff (x, y, u, v) \in D_2$$

extremal point in D_1 $\leftrightarrow$ extremal point in D_2

Fact: Finite intersections of convex sets is convex.

$$x, y \in D, \quad (1-t)x + t\cdot y \in D, \; 0 \le t \le 1.$$





$$\begin{aligned}
P=(2,\tfrac{4}{3}) &\longleftrightarrow (2, \tfrac{4}{3}, 0, 0)\\
(0,2) &\longleftrightarrow (0,2,0,6)\\
(0,0) &\longleftrightarrow (0,0,6,12)\\
(3,0) &\longleftrightarrow (3,0,3,0).
\end{aligned}$$

$$\boxed{\begin{cases} x+3y+u=6 \\ 4x+3y+v=12 \end{cases}} \quad x, y, u, v \ge 0.$$

$$\Updownarrow$$

$$\begin{pmatrix} 1 & 3 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

$$\begin{array}{cccc} \| & \| & \| & \| \\ v_1 & v_2 & v_3 & v_4 \end{array}$$

$$\| $$

$$x\cdot v_1 + y\cdot v_2 + u\cdot v_3 + v\cdot v_4$$

basic feasible solutions

$$\Updownarrow$$

extremal points of D_2

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_n \end{cases}$$

$\Updownarrow$

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} = b = Ax = (v_1\, v_2 \cdots v_n)\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \underline{x_1 v_1 + x_2 v_2 + \cdots + x_n \cdot v_n}$$

$$A = \left(\begin{array}{c|c|c|c} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{array}\right) = (v_1\, v_2 \cdots v_n) \qquad m \leq n.$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

Definition (basic solution) Assume that $v_{i_1}, \cdots, v_{i_m}$ are linearly independent. $\quad i_1 < i_2 < \cdots < i_m$

$$\Leftrightarrow \underset{m\times m}{B} = (v_{i_1}\, v_{i_2} \cdots v_{i_m}) \text{ is invertible}$$

basic variables.

$$\underset{m\times m}{B} \cdot x' = b \Rightarrow x' = B^{-1}b = (\underline{x_{i_1}, x_{i_2}, \cdots, x_{i_m}})$$

$$\rightsquigarrow x_B = (0 \cdots 0\, x_{i_1}\, 0 \cdots 0\, x_{i_2}\, 0 \cdots 0\, x_{i_m}\, 0 \cdots 0)$$

basic solutions.

non-basic

$$\begin{array}{c} \begin{array}{cccc} x & y & u & v \end{array} \\ \begin{pmatrix} 1 & 3 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{pmatrix} \end{array} \begin{pmatrix} x \\ y \\ u \\ v \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

2×4

	$B x_B = b$	x_B	BF
(x, y)	$\begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$	$(2, \frac{4}{3}, 0, 0) \leftarrow$	✓
(x, u)	$\begin{pmatrix} 1 & 1 \\ 4 & 0 \end{pmatrix}\begin{pmatrix} x \\ u \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$	$(3, 0, 3, 0) \leftarrow$	✓
(x, v)	$\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$	$(6, 0, 0, -12)$	
(y, u)	$\begin{pmatrix} 3 & 1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} y \\ u \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$	$(0, 4, -6, 0)$	
(y, v)	$\begin{pmatrix} 3 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} y \\ v \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$	$(0, 2, 0, 6) \leftarrow$	✓
(u, v)	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$	$(0, 0, 6, 12) \leftarrow$	✓

$$\binom{n}{m} = \frac{n!}{m!(n-m)!} = \frac{n(n-1)\cdots(n-m+1)}{m!}, \quad \binom{4}{2} = \frac{4!}{2!\,2!} = \frac{4 \times 3 \times \cancel{2 \times 1}}{(2 \times 1)\cancel{(2 \times 1)}} = \frac{4 \times 3}{2 \times 1} = 6,$$

$(2, \frac{4}{3}, 0, 0)$

$(0, 2, 0, 6)$

$(0, 0, 6, 12)$

$(3, 0, 3, 0)$.