Linear Programing Problem in Standard Form:

Maximize $z = c_1 x_1 + c_2 x_2 + \cdots + c_n x_n$

subject to

$$\text{m constraints}\begin{cases} a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le b_1 \\ a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le b_2 \\ \quad \cdots \\ a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le b_n \end{cases}$$

$x_i \geq 0, \quad i = 1, \cdots, n$

$$C = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \qquad C^T = (c_1 \; c_2 \; \cdots \; c_n)$$

$$\underset{m \times n}{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \qquad x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \qquad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

Maximize $z = (c_1 \; c_2 \; \cdots \; c_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = C^T \cdot X$

Subject to

$$\underset{m \times n}{A} \cdot \underset{n \times 1}{x} \le \underset{m \times 1}{b}$$

$$x \geq 0$$

Slack variable:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$

$$\Updownarrow$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + u_1 = b_1 \\ u_1 \geq 0. \end{cases}$$

$1 \leq i \leq m$

i-th $\quad a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$

$$\Updownarrow$$

$$\begin{cases} a_{i1}x_1 + a_{2i}x_2 + \cdots + a_{in}x_n + u_i = b_i \\ u_i \geq 0. \end{cases}$$

Maximize $\quad z = C^T \cdot x \qquad u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix}$

Subject to $\quad \underset{m\times n}{A}\ \underset{n\times 1}{x} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = b \Leftrightarrow Ax + u = b$

$$x \geq 0. \quad u \geq 0.$$

Canonical form of LP

$$a_1x_1 + \cdots + a_nx_n = b$$

$$\Updownarrow$$

$$\begin{cases} a_1x_1 + \cdots + a_nx_n \leq b \\ -a_1x_1 - \cdots - a_nx_n \leq -b. \end{cases}$$

Standard form $\longleftrightarrow$ Canonical form

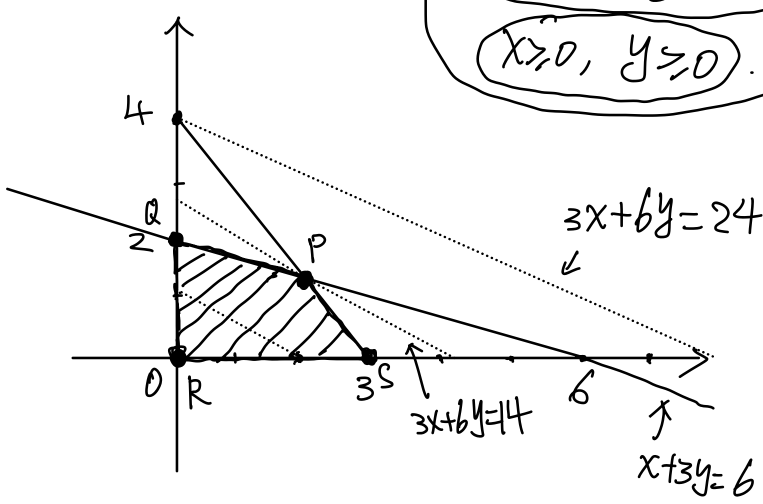
("$\leq$") ("$=$")

Ex: Maximize $z = 3x + 6y$

Subject to $x + 3y \leq 6 \rightarrow x+3y=6$

$4x + 3y \leq 12 \rightarrow 4x+3y=12$

$x \geq 0,\ y \geq 0.$



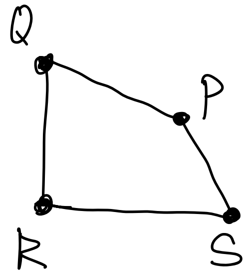
$3x + 6y = 6$

$x + 2y = 2$

$$\begin{cases} x+3y=6 \\ 4x+3y=12 \end{cases} \Rightarrow \quad 3x=6 \Rightarrow x=2 \quad \Rightarrow 3y=4 \Rightarrow y=\frac{4}{3}$$

$\Rightarrow$ Optimal solution at $(x,y)=(2,\frac{4}{3})$

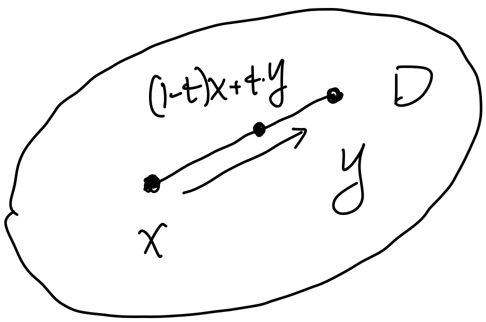
Maximum $3x+6y=3\cdot 2+6\cdot\frac{4}{3}=14.$



Extremal Points: $\{P, Q, R, S\}$.

Convex set: subset $D \subset \mathbb{R}^n$ that satisfies.

If $\underset{\substack{\parallel \\ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}}{x}$, $\underset{\substack{\parallel \\ \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}}}{y} \in D$, then the line segment from x to y are all contained in D.

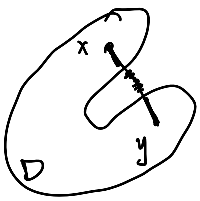


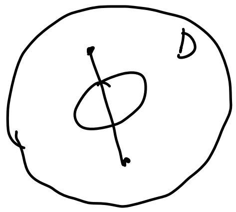
$$\{(1-t)x+\underbrace{ty}_{}\overset{x_t}{} : 0\le t\le 1\} \subseteq D.$$

where $x_t = (1-t)x+ty$

$t=0$: $x_0=x$.

$t=1$: $x_1=y$.

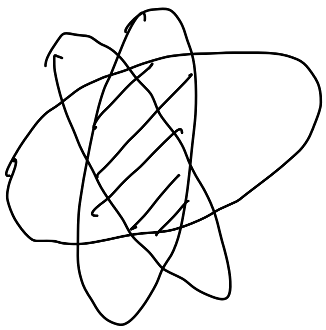




Facts: • Any half space is convex.

$\{(x_1,\cdots,x_n): a_1 x_1 + a_2 x_2 + \cdots + a_n x_n \le b_n\}$ is convex.

• Intersection of convex sets is convex.

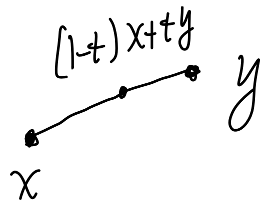


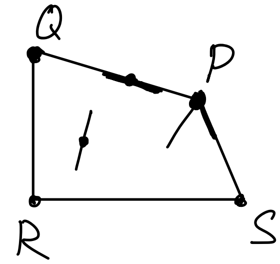
$\Rightarrow$ The regions defined by the constraints are always convex.

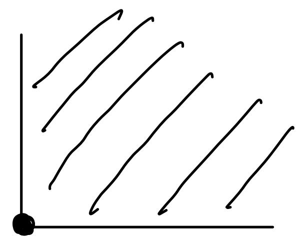
Def: A ~~point~~ u in convex set D is an extremal point if it is not an interior point of any line segment in D: There is no $x, y \in D$ s.t.

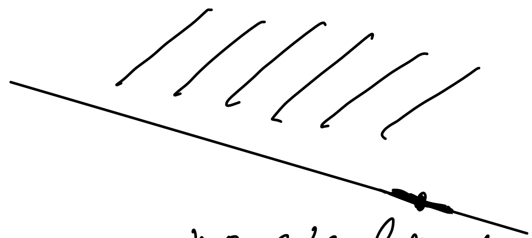
$$u = (1-t)x + ty \quad \text{with } 0 < t < 1.$$

u is extremal : $u \in \underset{\substack{\uparrow \\ \text{line seg.}}}{\ell} \subset D$, then u is one end point of this line seg.









no extremal point.

LP Problem :

$$x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$$\boxed{\begin{array}{ll} \text{Maximize} & z = c^T x \\ \text{subject to} & Ax \le b \\ & x \ge 0 \end{array}}$$

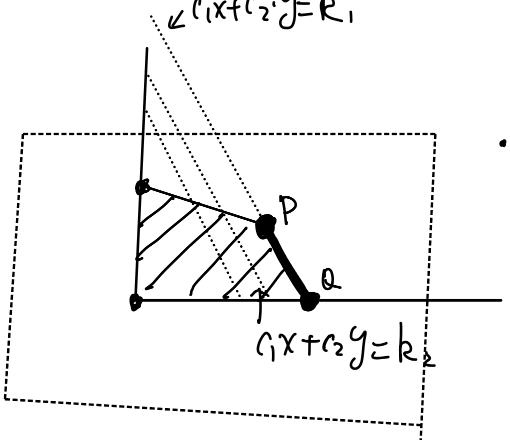
$\rightsquigarrow$ Region in $\mathbb{R}^n$ $=:$ set of feasible solution.

$$\mathbb{R}^n \supseteq D = \{x : Ax \le b,\ x \ge 0\}$$

$\Downarrow$

x is called an optimal solution if it solves this LPP.

<u>Thm</u> (Extremal Point Thm) For any LPP, if an optimal solution exists, then an optimal solution occurs at an extremal point.

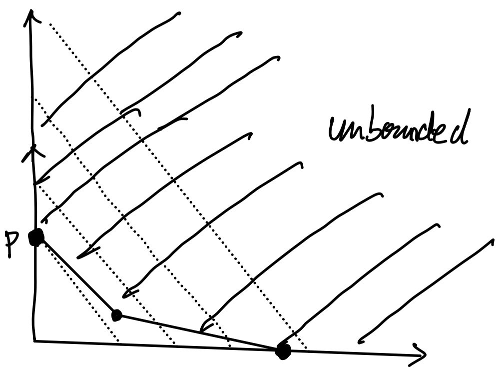


$Z = c_1 x + c_2 \cdot y$.

- When the feasible solution set is bounded, then there exists an optimal solution.

$\Downarrow$

an optimal solution occurs at an extremal point



- unbounded feasible sol. set

$\Downarrow$

Optimal solution may or may not exist.

- feasible solution set is empty.