(i) We need to prove that there is no nontrivial representation of $0$ as a linear combination of $S$. Assume 

$$a \cdot (x^2+1) + b \cdot (x^2+x) = 0.$$ 

Then $\begin{cases} a+b=0 \\ b=0 \\ a=0 \end{cases}$, so only trivial rep. of $0$.

(ii) $\beta = \{x^2+1, x^2+x, 1\}$ is a basis because that $\beta$ is linearly independent and $\# \beta = 3 = \dim P_2(\mathbb{R})$.

If $a(x^2+1)+b(x^2+x)+c \cdot 1 = 0$ then $
\begin{cases} a+b=0 \\ b=c=0 \end{cases}$ or $a+c=0$.

OR: $\beta$ spans $P_2(\mathbb{R})$ and $\# \beta = 3$:

$1 = 1$, $x = x^2+x-(x^2+1)+1$, $x^2 = (x^2+1)-1$. 
2(20pts) Consider the linear transformation:

\[ T : P_2(\mathbb{R}) \to M_{2 \times 2}(\mathbb{R}), \quad T(f) = \begin{pmatrix} f(0) & f'(0) \\ f(1) & f'(1) \end{pmatrix}. \]

(1) Find the matrix representation of \( T \) with respect to the bases \( \beta = \{1, x, x^2\}, \quad \gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}. \)

Let \( f(x) = 3 - 2x + x^2 \). Calculate \([f(x)]_\beta\) and \([T(f(x))]_\gamma\).

(1) \[ f = a + bx + c \cdot x^2 \quad \Rightarrow \quad f(0) = a, \quad f'(0) = b, \quad f(1) = a + b + c, \quad f'(1) = b + 2c \]

\[ \Rightarrow \quad T(f) = \begin{pmatrix} a & b \\ a + b + c & b + 2c \end{pmatrix} \]

\[ [T]_\gamma = \begin{pmatrix} [T(1)]_\gamma & [T(x)]_\gamma & [T(x^2)]_\gamma \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \]

(2) \[ [f(x)]_\beta = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \]

\[ [T(f(x))]_\gamma = [T]_\gamma \cdot [f(x)]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix} \]

OR \[ T(f(x)) = \begin{pmatrix} 3 & -2 \\ 2 & 0 \end{pmatrix} \Rightarrow \quad [T(f(x))]_\gamma \]
3 (20 pts) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined as:

$$T(x, y, z) = (y - 2z, x - 2z, x - y).$$

Calculate nullity($T$) and rank($T$).

$$N(T) = \left\{ (x, y, z) \in \mathbb{R}^3 : T(x, y, z) = 0 \right\}$$

$$0 = T(x, y, z) = (y - 2z, x - 2z, x - y) \iff \begin{cases} y = 2z \\ x = 2z \\ x = y \end{cases}$$

$$\forall z \in \mathbb{R}, \quad (x, y, z) = (2z, 2z, z) = z \cdot (2, 2, 1)$$

So, $N(T) = \text{Span}\{ (2, 2, 1) \}$.

$$\text{nullity}(T) = \dim N(T) = 1.$$  \[\boxed{8}\]

$$\text{rank}(T) = \dim \mathbb{R}^3 - \text{nullity}(T) = 3 - 1 = 2.$$  \[\boxed{8}\]
Let $S$ be a linearly independent subset of $V$. Prove that $T(S) = \{T(v_1), \ldots, T(v_r)\}$ is a linearly independent subset of $W$. 

4.(40 pts) Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces. Assume that $T$ is one-to-one but not onto.

(2) (10 pts) Prove the inequality $\dim V < \dim W$.

(3) (10 pts) Prove that there exists a linear transformation $U : W \to V$ such that $UT = 1_{V}$ (Hint: define $U$ by its values on basis vectors).

(4) (10 pts) Is there a linear transformation $U : W \to V$ such that $TU = 1_{W}$? Explain your reasons.

Proof: (1) Assume $a_1 T(v_1) + a_2 T(v_2) + \cdots + a_r T(v_r) = 0$.

Then $T(a_1 v_1 + a_2 v_2 + \cdots + a_r v_r) = 0$.

Because $T$ is one-to-one, $a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0$.

Because $\{v_1, \ldots, v_r\}$ is linearly independent, $a_1 = a_2 = \cdots = a_r = 0$.

So there is only trivial linear relation for $T(S)$, $T(S)$ is linearly independent.

(2) Choose a basis $\beta = \{v_1, v_2, \ldots, v_n\}$ for $V$. $\beta$ is linearly independent.

By part (1), $T(\beta) = \{T(v_1), T(v_2), \ldots, T(v_n)\}$ is linearly independent.

So $\#(T(\beta)) = n \leq \dim W$ because $T(\beta)$ can be extended to a basis $Y$ for $W$.

$\dim V = \#(T(\beta)) < \# Y = \dim W$.

Because $\dim V < \dim W$, $Y \neq T(\beta)$. Otherwise $\text{span}(T(\beta)) = \text{span}(\beta)$.

So $\dim V = \#(T(\beta)) < \# Y = \dim W$.

$T$ is not onto.
Continuation of works:

(3) Continuing with the notation from part (2), assume:

\[ y = \{ T(v_1), T(v_2), \ldots, T(v_n), w_{n+1}, \ldots, w_m \} \text{ with } n < m \]

Define the linear transformation \( U \) by setting:

\[ U(w_i) = v_i, \ldots, U(w_n) = v_n, \ U(w_{n+1}) = 0, \ldots, U(w_m) = 0. \]

Then \( UT(v_i) = U(w_i) = v_i \), for \( 1 \leq i \leq n \).

So \( UT = Id_v \) (because \( \beta = \{ v_1, \ldots, v_n \} \) is a basis).

(4) If \( TU = Id_v \), then \( T \) is onto, which contradicts the assumption that \( T \) is not onto. So there is no such \( U \).