HW 3:

1.5 2(e): Determine linearly dependent/independent:

(e) \( \{ (1,-1,2), (1,-2,1), (1,1,4) \} \) in \( \mathbb{R}^3 \)

Determine whether it is linearly independent: Find relations \( \sum \) representations of \( \mathbf{0} \).

\[
\begin{pmatrix}
0 \\
0
\end{pmatrix} = a_1 \begin{pmatrix}
1 \\
-1
\end{pmatrix} + a_2 \begin{pmatrix}
2 \\
1
\end{pmatrix} + a_3 \begin{pmatrix}
1 \\
4
\end{pmatrix} = \begin{pmatrix}
1 \\
-2
\end{pmatrix} + a_3 \begin{pmatrix}
2 \\
4
\end{pmatrix}
\]

\[
\begin{pmatrix}
1 & 1 & 1 \\
-2 & 1 & 4
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 1 & 0 \\
0 & 1 & -2
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} \rightarrow \begin{pmatrix}
1 & 0 & 3 \\
0 & 1 & -2
\end{pmatrix} \begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} \Rightarrow \begin{pmatrix}
a_1 = -3a_3 \\
a_2 = 2a_3
\end{pmatrix}
\]

There are nonzero solutions of \( \begin{pmatrix} a_3 \\ a_2 \\ a_1 \end{pmatrix} \) (because there is a free variable \( a_3 \))

\[ \Rightarrow \text{ there are nontrivial rep. of } \mathbf{0} \Rightarrow \text{ linearly dependent.} \]

\[
\begin{pmatrix}
-3 \\
2
\end{pmatrix} = a_1 \begin{pmatrix}
1 \\
-2
\end{pmatrix} + a_2 \begin{pmatrix}
2 \\
1
\end{pmatrix} + a_3 \begin{pmatrix}
1 \\
4
\end{pmatrix} = \begin{pmatrix}
-3 + 2 + 1 \\
-6 + 2 + 4
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

15.9, \( u \neq v \in V \). Prove: \( \{u, v\} \) is linearly dependent \( \Leftrightarrow \) \( u \) or \( v \) is a multiple of the other.

Proof: 

"if" \( \Rightarrow \) \( u = c \cdot v \), then \((-1)u + c \cdot v = 0 \Rightarrow \text{linearly dependent.} \)

"only if" \( \iff \{u, v\} \text{ is linearly dependent, then } \exists (a, b) \neq (0, 0) \text{ s.t.} \)

\[ au + bv = 0 \text{.} \]

two cases \( a \neq 0 \): then \( u = -a^{-1} b \cdot v \), \( u \) is a multiple of \( v \)

\( b \neq 0 \): then \( v = -b^{-1} a \cdot u \), \( v \) is a multiple of \( u \)
1.5.15 \[ S = \{ u_1, u_2, \ldots, u_n \} \]

Prove \( S \) is linearly dependent if \( u_i = 0 \) or \( u_{k+1} \in \text{Span}\{u_1, \ldots, u_k\} \) for some \( 1 \leq k < n \).

**Proof:** "if" \( u_i = 0 \), then \( S \) is linearly dependent: 0 = \( 1 \cdot u_i \)

\[ \begin{align*}
\text{if } u_{k+1} &\in \text{Span}\{u_1, \ldots, u_k\}, \text{ then } u_{k+1} = c_1 u_1 + \cdots + c_k u_k, 1 \leq k < n. \\
\Rightarrow (c_1) u_{k+1} + c_1 u_1 + \cdots + c_k u_k = 0 \Rightarrow S \text{ is linearly dependent}
\end{align*} \]

"only if": Assume \( S \) is linearly dependent. Then \( \exists a_1, \ldots, a_n \) \begin{enumerate}
\item[1.] not all zero
\item Let \( m = \text{max} \{ i \in \{1, \ldots, n\} \text{ such that } a_i \neq 0 \} \)
\item If \( m = 1 \), then \( 0 = a_1 u_1 + u_2 + \cdots + u_n = u_1 \Rightarrow u_1 = 0 \)
\item If \( m > 1 \), then \( 0 = a_1 u_1 + a_2 u_2 + \cdots + a_m u_m + 0 u_{m+1} + \cdots + 0 u_n \)
\item \( u_m = -a_m^{-1} a_1 u_1 - a_m^{-1} a_2 u_2 - \cdots - a_m^{-1} a_{m-1} u_{m-1} \)
\item \( u_{m+1} \in \text{Span}\{u_1, \ldots, u_m\} \) let \( k = m-1 \in \{1, \ldots, n-1\} \)
\end{enumerate}

1.6.3(a): Determine whether the subset is a basis for \( P_2(\mathbb{R}) \): \[ \{-1-x+2x^2, 2+x-2x^2, 1-2x+4x^2\} \]

Just need to determine if it is linearly dependent or not.

Find representation of 0 polynomial:
\[ 0 = a_1 (-1-x+2x^2) + a_2 (2+x-2x^2) + a_3 (1-2x+4x^2) \]
\[ \Rightarrow \begin{cases} 
-a_1 + 2a_2 + a_3 = 0 \\
-a_1 + a_2 - 2a_3 = 0 \\
2a_1 - 2a_2 + 4a_3 = 0 
\end{cases} \]
\[
\begin{pmatrix}
-1 & 2 & 1 \\
-1 & 1 & -2 \\
2 & -2 & 4
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0
\end{pmatrix}
\]

\[
\begin{pmatrix}
-1 & 2 & 1 \\
-1 & 1 & -2 \\
2 & -2 & 4
\end{pmatrix}
\overset{(1) \cdot (1) + (2) \cdot 0}{\overset{0 \cdot 2 \cdot 0}{\rightarrow}}
\begin{pmatrix}
-1 & 2 & 1 \\
0 & -1 & -3 \\
0 & 0 & 6
\end{pmatrix}
\overset{(1) \cdot 2 \cdot -1}{\rightarrow}
\begin{pmatrix}
1 & 2 & -1 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{pmatrix}
\overset{0 \cdot 0 \cdot 0}{\rightarrow}
\begin{pmatrix}
1 & 0 & 5 \\
0 & 1 & 3 \\
0 & 0 & 0
\end{pmatrix}
\]

\[
\begin{cases}
a_1 = -5a_3 \\
a_2 = -3a_3
\end{cases} \implies \exists \text{ non-zero solutions (when } a_3 \neq 0, \text{ e.g. } \left( \frac{-5}{3} \right) \)
\]

So the set is not linearly independent and is not a basis.

3.(b) \{1+2x+x^2, 3x+x^2, x+x^2 \}. Similar method:

\[
\begin{pmatrix}
a_1 \cdot (1+2x+x^2) + a_2 \cdot (3x+x^2) + a_3 \cdot (x+x^2) = 0
\end{pmatrix} \iff
\begin{cases}
a_1 + 3a_2 = 0 \\
a_2 + 3a_3 = 0 \\
a_1 + a_2 + a_3 = 0
\end{cases}.
\]

\[
\begin{pmatrix}
1 & 3 & 0 \\
2 & 0 & 1 \\
1 & 1 & 1
\end{pmatrix}
\overset{0 \cdot 1 \cdot 0}{\rightarrow}
\begin{pmatrix}
1 & 3 & 0 \\
0 & -6 & 1 \\
0 & -2 & 1
\end{pmatrix}
\overset{0 \cdot 0 \cdot -2}{\rightarrow}
\begin{pmatrix}
1 & 3 & 0 \\
0 & 2 & -1 \\
0 & 0 & 2
\end{pmatrix}
\]

So the set is linearly independent. Only zero solutions for \( \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \)

Because it consists of 3 polynomials, by [Corollary 2(b), pg. 48],

\[\dim P_2(\mathbb{R}) \]

it is a basis for \( P_2(\mathbb{R}) \).
15. \[ W = \{ A \in M_{n \times n}(F) : \text{trace}(A) = 0 \} \]

\[ A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} \in W \iff a_{11} + a_{22} + \cdots + a_{nn} = 0 \]

\[ a_{nn} = -a_{11} - a_{22} - \cdots - a_{n-1,n-1} \]

\[ \Rightarrow A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & -a_{11} - a_{22} - \cdots - a_{nn}
\end{pmatrix} \]

Basis vectors:

\[ E_{i,j} = \begin{pmatrix}
    0 & \cdots & 0 & \cdots & 0 \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    0 & \cdots & 0 & \cdots & 0 \\
    \cdots & \ddots & \ddots & \ddots & \vdots \\
    0 & \cdots & 0 & \cdots & 0
\end{pmatrix}_{i \neq j} \]

\[ E_{11} - E_{nn} = \begin{pmatrix}
    1 & 0 & \cdots & 0 \\
    0 & 1 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 0
\end{pmatrix} \]

\[ E_{22} - E_{nn} = \begin{pmatrix}
    0 & 1 & \cdots & 0 \\
    1 & 0 & \cdots & 0 \\
    \vdots & \vdots & \ddots & \vdots \\
    0 & 0 & \cdots & 0
\end{pmatrix} \]

\[ E_{n-1,n-1} - E_{nn} = \begin{pmatrix}
    0 & \cdots & 0 \\
    \vdots & \ddots & \vdots \\
    0 & \cdots & 0 \\
    \cdots & \ddots & \ddots \\
    0 & \cdots & 0
\end{pmatrix} \]

\[ \dim = n(n-1) + (n-1) = n^2 - 1 \]
21. Prove that a vector space is infinite dimensional iff it contains an infinite linearly independent subset.

Proof: A vector space is called infinite dimensional if it has no basis consisting of finite number of vectors. (Definitions on page 47)

"iff" Assume $V$ contains an infinite linearly independent subset.

If $V$ contains a basis $\beta$ with $\#\beta = \dim V < \infty$, then by the Replacement Theorem, any linearly independent subset has at most $\#\beta = \dim V$ vectors (since $V$ is generated by $\beta$). This contradicts the assumption. So $V$ must be infinite dimensional.

"only if" $V$ is infinite dimensional. We can construct an infinite linearly independent subset in the following way by induction:

- **Step 1**: Choose a nonzero $v \in V$. Set $S_1 = \{v\}$.

- **Step $n \Rightarrow$ Step $n+1$**: Assume we have a linearly independent subset $S_n = \{v_1, \ldots, v_n\}$ of size $n$.

Then $S_n$ can not span $V$ because otherwise $S_n$ contains a basis of $V$ by [Theorem 1.9, pg. 45]. So there exists $v_{n+1} \in V$ s.t. $v_{n+1} \notin \text{Span}(S_n)$. By [Theorem 1.7, pg. 41], the subset $S_{n+1} = S_n \cup \{v_{n+1}\}$ is linearly independent.

So in this way, we can construct an infinite linearly independent subset $\{v_i : i \in \mathbb{N}\}$. 
26. Subspace of $P_n(\mathbb{R})$ \( W = \{ f \in P_n(\mathbb{R}) : f(a) = 0 \} \).

$\exists f = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n$ with $b_0, \ldots, b_n \in \mathbb{R}$ is contained in $W$ if and only if

\[
\begin{align*}
&b_0 + b_1 a + b_2 a^2 + \cdots + b_n a^n = 0 \\ &\iff b_0 = -b_1 a - b_2 a^2 - \cdots - b_n a^n \\ &\iff f = (-b_1 a - b_2 a^2 - \cdots - b_n a^n) + b_1 x + \cdots + b_n x^n \\ &\quad = b_1 (-a+x) + b_2 (-a^2+x^2) + \cdots + b_n (-a^n+x^n)
\end{align*}
\]

so \( W = \text{Span} \beta \) where

\[
\beta = \{-a+x, -a^2+x^2, \ldots, -a^n+x^n\}
\]

It is easy to show that $\beta$ is linearly independent (by using the fact the polynomials in $\beta$ are of different degrees).

So $\beta$ is a basis for $W$ and $\dim W = n$.

$\dim P_n(\mathbb{R}) = \frac{(n+1)(n)}{2} - 1$. 