T: $V \to V$ linear transformation

Def: A subspace $W$ of $V$ is called a $T$-invariant subspace if $T(W) \subseteq W$, i.e., $T(v) \in W$ for any $v \in W$.

Examples: \( \{0\}, V, N(T) \).

$R(T) := T(R(T)) \subseteq R(T)$.

$E_\lambda = N(T-\lambda I): \forall v \in N(T-\lambda I) \implies T(v) = \lambda \cdot v \implies Tv \in W.$

Def: Fix any $v \in V$. The $T$-cyclic space generated by $v$ is

the subspace $\text{Span}\{v, Tv, T^2v, \ldots\} = \text{Span}\{T^k v; k=1,2,\ldots\}$

Then: If $W = \text{Span}\{v, Tv, T^2v, \ldots\}$ has dimension $m$, then

$\{v, Tv, \ldots, T^{m-1}v\}$ is a basis for $W$.

Proof: It suffices to prove the following statement: Let $m$ be the largest integer such that $\{v, Tv, \ldots, T^{m-1}v\}$ is linearly independent. Then $\beta$ is a basis for $W$.

Proof: We only need to show $\beta$ spans $W$.

Actually it is enough to show that $T^m v$ is a linear combination of $\beta$ because we can then iteratively write $T^k v$ as a linear combination of $\beta$. 
for any \( k \geq m \):
\[
T^k m = T^{k-m} (T^m V) = \ldots .
\]
For \( k = m \), because the subset \( \{ v, TV, T^2v, \ldots, T^{m-1}v, T^m v \} \) is linearly dependent by the definition of \( m \), we know that there is a non-trivial linear relation:
\[
a_0 v + a_1 TV + a_2 T^2v + \ldots + a_{m-1} T^{m-1}v + a_m T^m v = 0
\]
where not \( \{ a_i, i = 0, \ldots, m \} \) are not all zero.
We must have \( a_m \neq 0 \) since \( \{ v, TV, \ldots, T^{m-1}v \} \) is linearly independent.
So
\[
T^m v = -a_{m-1} a_1 T^1 v - a_{m-2} a_1 TV - \ldots - a_1 a_1 T^m v
\]
as wanted.