$$A=\begin{pmatrix}8&-2&2\\-2&5&4\\2&4&5\end{pmatrix}.$$ orthogonal diagonalization.

$-\frac{1}{2}(\lambda^2-13\lambda+36)$

$\parallel$

$-\frac{8-\lambda}{2}\cdot 4-2=-2(9-\lambda)$

$-\frac{8-\lambda}{2}(5-\lambda)+2=-\frac{1}{2}(\lambda^2-13\lambda+40)+2$

1. $$|A-\lambda I|=\begin{vmatrix}8-\lambda&-2&2\\-2&5-\lambda&4\\2&4&5-\lambda\end{vmatrix}=\begin{vmatrix}0&\downarrow&\leftarrow\\0&9-\lambda&9-\lambda\\2&4&5-\lambda\end{vmatrix}$$

$$=-2\cdot(9-\lambda)\cdot\begin{vmatrix}+2(9-\lambda)&+\frac{1}{2}(\lambda^2-13\lambda+36)\\1&1\end{vmatrix}=-2\cdot(9-\lambda)\cdot\left(2(9-\lambda)-\frac{1}{2}(\lambda^2-13\lambda+36)\right)$$

$$=-(9-\lambda)\left[-\lambda^2+9\lambda\right]=-\lambda(\lambda-9)^2=0\Rightarrow \lambda=0,\ \text{mult}(0)=1.$$
$$\lambda=9,\ \text{mult}(9)=2.$$

2

2. $\lambda=0$: $$A-0\cdot I=\begin{pmatrix}8&-2&2\\-2&5&4\\2&4&5\end{pmatrix}\to\begin{pmatrix}0&-18&-18\\0&9&9\\2&4&5\end{pmatrix}\to\begin{pmatrix}2&4&5\\0&1&1\\0&0&0\end{pmatrix}\to\begin{pmatrix}2&0&1\\0&1&1\\0&0&0\end{pmatrix}$$

$$\Rightarrow E_0=N(A-0I)=\text{Span}\left\{\underbrace{\begin{pmatrix}-1\\-2\\2\end{pmatrix}}_{=w_1}\right\}$$

2

$\lambda=9$: $$A-9I=\begin{pmatrix}-1&-2&2\\-2&-4&4\\2&4&-4\end{pmatrix}\to\begin{pmatrix}1&2&-2\\0&0&0\\0&0&0\end{pmatrix}\Rightarrow E_9=\text{Span}\left\{\underbrace{\begin{pmatrix}-2\\1\\0\end{pmatrix}}_{w_2},\underbrace{\begin{pmatrix}2\\0\\1\end{pmatrix}}_{w_3}\right\}$$

3. Note that $E_0\perp E_9$. $v_1=w_1$, $\|v_1\|^2=1^2+2^2+2^2=9$

Gram-Schmidt to $\{w_2,w_3\}$

$v_2=w_2=\begin{pmatrix}-2\\1\\0\end{pmatrix}$, $\|v_2\|^2=\langle v_2,v_2\rangle=2^2+1^2=5$

$\langle w_3,v_2\rangle=-4$

$$v_3=w_3-\frac{\langle w_3,v_2\rangle}{\|v_2\|^2}v_2=\begin{pmatrix}2\\0\\1\end{pmatrix}-\frac{-4}{5}\begin{pmatrix}-2\\1\\0\end{pmatrix}=\frac{1}{5}\cdot\begin{pmatrix}2\\4\\5\end{pmatrix}$$

3

$$\|v_3\|^2=\frac{1}{25}\cdot(2^2+4^2+5^2)=\frac{1}{25}\cdot 45=\left(\frac{3\sqrt{5}}{5}\right)^2$$

$$\Rightarrow u_1=\frac{v_1}{\|v_1\|}=\frac{1}{3}\cdot\begin{pmatrix}-1\\-2\\2\end{pmatrix},\quad u_2=\frac{v_2}{\|v_2\|}=\frac{1}{\sqrt5}\begin{pmatrix}-2\\1\\0\end{pmatrix},\quad u_3=\frac{v_3}{\|v_3\|}=\frac{1}{3\sqrt5}\cdot\begin{pmatrix}2\\4\\5\end{pmatrix}$$

$$\Rightarrow S=(u_1\ u_2\ u_3)=\begin{pmatrix}-\frac{1}{3}&\frac{-2}{\sqrt5}&\frac{2}{3\sqrt5}\\-\frac{2}{3}&\frac{1}{\sqrt5}&\frac{4}{3\sqrt5}\\\frac{2}{3}&0&\frac{5}{3\sqrt5}\end{pmatrix}$$ satisfies $$S^TAS=\begin{pmatrix}0&0&0\\0&9&0\\0&0&9\end{pmatrix}.$$