!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : ID :

1(25pts) A is a 3×5 matrix whose reduced row ethelon form is equal to:

$$\begin{pmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

(1) Find a basis for the null space $N(A)$.
(2) Assume that $A = (v_1\ v_2\ v_3\ v_4\ v_5)$ where v_i denotes the i-th column.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Recover the matrix A. (Be careful of the subscripts of column indices)

(1) $x \in \mathbb{R}^5$ is in $N(A) \Leftrightarrow Ax = 0 \Leftrightarrow rref(A)x = 0$ 5

$$\Leftrightarrow \begin{cases} x_1 - 2x_3 - x_5 = 0 \\ x_2 + x_3 + x_5 = 0 \\ x_4 + 2x_5 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 2x_3 + x_5 \\ x_2 = -x_3 - x_5 \\ x_4 = -2x_5 \end{cases}$$

basis for $N(A)$

$$\Leftrightarrow x = \begin{pmatrix} 2x_3 + x_5 \\ -x_3 - x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \in \text{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

5 5

(2) From $rref(A)$, we get the linear relations:

$v_3 = -2v_1 + v_2 \Rightarrow v_2 = v_3 + 2\cdot v_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ 5

$v_5 = -v_1 + v_2 + 2v_4 \Rightarrow v_5 = -\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + 2\cdot\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ 5

So $A = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 1 & 4 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$

2(25pts) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be the linear transformation given by

$$T(f) = x^2 f''(x) + f'(x) + 2f(x).$$

Let $\beta = \{1, x, x^2\}$ be the standard basis.

(1) Calculate the matrix representation $A = [T]_\beta$ w.r.t. the standard basis β.

(2) Determine whether T is diagonalizable or not. If yes, find Q such that $Q^{-1}AQ$ is a diagonal matrix.

(1) $$A = \left(\underset{\underset{[2]_\beta}{\parallel}}{[T(1)]_\beta} \ \underset{\underset{[1+2x]_\beta}{\parallel}}{[T(x)]_\beta} \ \underset{\underset{\underset{[2x+4x^2]_\beta}{\parallel}}{[x^2 \cdot 2 + 2x + 2x^2]_\beta}}{\parallel}{[T(x^2)]_\beta} \right) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

5 5

(2) Find eigenvalues of T (or equivalently for A).

multiplicity

$$A - \lambda I = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)^2 \cdot (4-\lambda) = 0 \Rightarrow \begin{matrix} \lambda = 2, \ m_2 = 2 \\ \lambda = 4, \ m_4 = 1 \end{matrix}$$

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Calculate $\dim E_2 = \dim N(A - 2I)$:

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim E_2 = \#\text{free var.} = 1$$

5

$1 = \dim E_2 < m_2 = 2 \Rightarrow A$ is not diagonalizable

$\Leftrightarrow T$ is not diagonalizable.

5

3(25pts) Let V be the subspace of $P_3(\mathbf{R})$ spanned by

$$1+x+x^2, \quad 2x+x^3, \quad 2+2x^2-x^3, \quad 1+x^3$$

(1) Find a basis for V.
(2) Is $f(x) = 2-x^2+x^3$ in the subspace V? Write down reason and calculations.

choose standard basis $\beta = \{1, x, x^2, x^3\}$. 4

$$A = \left([1+x+x^2]_\beta \; [2x+x^3]_\beta \; [2+2x^2-x^3]_\beta \; [1+x^3]_\beta \right) = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} = A.$$

Under the linear transformation $f(x) \mapsto [f(x)]_\beta$, $P_3(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$.

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V is isomorphic to the column space of A.

$2-x^2+x^3$ is in $V \Leftrightarrow \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{=b}$ is in the column space of A.

Solve (1) and (2) together:

$$\underbrace{\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right)}_{(A|b)} \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 0 & 2 & -2 & -1 & -2 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 & -4 \end{array}\right)$$

2

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right)$$

$\Rightarrow$ leading 1's are in 1st, 3rd, 4-th column.

$\Rightarrow$ (1) basis for V: $\{1+x+x^2, 2+2x^2-x^3, 1+x^3\}$. 6

(2) $\text{rank}(A|b) = 4 > \text{rank}(A) = 3 \Rightarrow b$ is not in the column space of A

$\Rightarrow 2-x^2+x^3$ is not in the subspace V. 5

6(20pts) Assume that V is a vector space with a basis $\beta = \{v_1, v_2, v_3, v_4\}$. Let $T : V \to V$ be a linear transformation that satisfies:

$$Tv_1 = v_2, \quad Tv_2 = v_3, \quad Tv_3 = v_4, \quad Tv_4 = 2v_2 + v_3.$$

(1) Calculate the characteristic polynomial of T.

(2) The linear transformation T satisfies the identity:

$$T^4 - T^2 = a \cdot T + b \cdot \mathrm{Id}_V$$

where Id_V is the identity transformation of V. Find the numbers a and b.

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(1). $[T]_\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = A$

$|A-\lambda I| = \begin{vmatrix} -\lambda & 0 & 0 & 0 \\ 1 & -\lambda & 0 & 2 \\ 0 & 1 & -\lambda & 1 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} -\lambda & 0 & 2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = -\lambda \cdot \begin{vmatrix} 0 & -\lambda^2 & \lambda+2 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix}$

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$= (-\lambda)\cdot(-1)\cdot \begin{vmatrix} -\lambda^2 & \lambda+2 \\ 1 & -\lambda \end{vmatrix} = \lambda\cdot(\lambda^3 - (\lambda+2)) = \underline{\lambda^4 - \lambda^2 - 2\lambda}$.

Characteristic polynomial of T.

(2) By Cayley-Hamilton Theorem,

$T^4 - T^2 - 2T = 0 \Leftrightarrow T^4 - T^2 = 2\cdot T + 0 \cdot \mathrm{Id}_V$ 5

5 so $a = 2,\ b = 0$.