MATH 350 FALL 2025 MIDTERM I

NAME: **ID:**

THERE ARE FOUR (4) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		20pts
2		30pts
3		20pts
4		30pts
Total		100pts

!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(20pts) Let $S = \{v_1, v_2, v_3\}$ be a subset of $\mathbf{R}^3$ with

$$v_1 = (1, -1, 1), v_2 = (1, 0, 1), v_3 = (1, 1, 1).$$

(1) Is S is a basis for $\mathbf{R}^3$? Why?

(2) Is there a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ that satisfies the following three equalities?

$$T(v_1) = (1, 0, 0), \quad T(v_2) = (0, 1, 0), \quad T(v_3) = (0, 1, 0).$$

Explain your reason.

(1) $V_3 = (1,1,1) = -(1,-1,1) + 2\cdot(1,0,1) = -V_1 + 2V_2$ 10

$\Rightarrow S = \{V_1, V_2, V_3\}$ is linearly dependent $\Rightarrow$ S is not a basis for $\mathbb{R}^3$

(2) $V_3 = -V_1 + 2V_2$

$\Rightarrow TV_3 = T(-V_1 + 2V_2) = -T(V_1) + 2\cdot T(V_2)$ for any linear transformation

If $T(V_1) = (1,0,0)$, $T(V_2) = (0,1,0)$, then

$-T(V_1) + 2T(V_2) = -(1,0,0) + 2\cdot(0,1,0) = (-1,2,0) \neq T(V_3) = (0,1,0)$

$\Rightarrow$ No such linear transformations. 10

2(30pts) Consider the linear transformation:

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R}), \quad T(f) = xf'(x).$$

Let $\beta = \{1, x, x^2\}$, $\gamma = \{1, 1+x, 1+x^2\}$ be two basis for $P_2(\mathbf{R})$.

(1) Find the matrix representation $[T]_\beta^\beta$.
(2) Find the matrix representation $[T]_\beta^\gamma$.
(3) Calculate $[T \circ T]_\beta^\gamma$.

(1) $[T]_\beta^\beta = \left(\underset{\underset{[0]_\beta}{\|}}{[T(1)]_\beta} \quad \underset{\underset{[x\cdot 1]_\beta}{\|}}{[T(x)]_\beta} \quad \underset{\underset{[x\cdot 2x]_\beta}{\|}}{[T(x^2)]_\beta} \right) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 10

(2) $[T]_\beta^\gamma = \left(\underset{\underset{[0]_\gamma}{\|}}{[T(1)]_\gamma} \quad \underset{\underset{[x]_\gamma}{\|}}{[T(x)]_\gamma} \quad \underset{\underset{[2x^2]_\gamma}{\|}}{[T(x^2)]_\gamma} \right) = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ 10

$\beta \ni 1 = 1\cdot 1 + 0\cdot(1+x) + 0\cdot(1+x^2) \Rightarrow [1]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$x = -1\cdot 1 + 1(1+x) + 0\cdot(1+x^2) \Rightarrow [x]_\gamma = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$x^2 = -1\cdot 1 + 0\cdot(1+x) + 1\cdot(1+x^2) \Rightarrow [x^2]_\gamma = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

(3) $[T\circ T]_\beta^\gamma = [T]_\beta^\gamma [T]_\beta^\beta = \begin{pmatrix} 0 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$

<u>OR</u>: $\underset{\underset{T^2(f(x))}{\|}}{(T\circ T)(f(x))} = T(xf'(x)) = x\cdot(xf'(x))' = x\cdot(f'(x) + x\cdot f''(x)) = x\cdot f'(x) + x^2 f''(x)$

$[T^2]_\beta^\gamma = \left(\underset{\underset{[0]_\gamma}{\|}}{[T^2(1)]_\gamma} \quad \underset{\underset{[x]_\gamma}{\|}}{[T^2(x)]_\gamma} \quad \underset{\underset{\underset{[4x^2]_\gamma}{\|}}{[x\cdot 2x + x^2\cdot 2]_\gamma}}{\|}}{[T^2(x^2)]_\gamma} \right) = \begin{pmatrix} 0 & -1 & -4 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ 10

3(20 pts) Let $T : P_3(\mathbf{R}) \to P_3(\mathbf{R})$ (NOT $P_2(\mathbf{R})$) be the linear transformation defined as:

$$T(f(x)) = xf'(x) - f(x).$$

(1) What is the null space $N(T)$ of T?
(2) What is the rank of T?

(1) $f \in N(T) \Leftrightarrow T(f(x)) = x \cdot f'(x) - f(x) = 0$

If $f(x) = ax^3 + bx^2 + c \cdot x + d$, then

$T(f(x)) = x \cdot (3ax^2 + 2bx + c) - (ax^3 + bx^2 + cx + d)$

$= 2ax^3 + bx^2 - d$

So $T(f(x)) = 0 \Leftrightarrow a = b = d = 0 \Leftrightarrow f(x) = c \cdot x$ for any $c \in \mathbb{R}$

$\Rightarrow N(T) = \{c \cdot x . c \in \mathbb{R}\} = \text{Span}\{x\}$.

10

(2) From (1), $\text{null}(T) = \dim N(T) = 1$.

So $\text{rank}(T) = \dim V - \text{null}(T) = 4 - 1 = 3$.

$V = P_3(\mathbb{R}) = \text{Span}\{1, x, x^2, x^3\}$

10

4(30pts) Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces.

(1) Assume that there a linear transformation $S : W \to V$ such that $S \circ T = \mathrm{Id}_V$. Must T be one-to-one or onto? Must S be one-to-one or onto? Prove your claims.

(2) Assume that T is one-to-one. Prove that $\dim V \le \dim W$.

(1) In general, we have:

$$N(S \circ T) \supseteq N(T), \quad R(S \circ T) \subseteq R(S)$$

$$S \circ T = \mathrm{Id}_V \Rightarrow \{0\} = N(\mathrm{Id}_V) \supseteq N(T) \Rightarrow N(T) = \{0\}$$ 10

$$R(\mathrm{Id}_V) = V \subseteq R(S) \Rightarrow R(S) = V$$ 10

$\Rightarrow$ T must be one-to one and S must be onto.

(2) T is one-to-one $\Leftrightarrow N(T) = \{0\} \Leftrightarrow \mathrm{null}(T) = 0$. 10

$$\Rightarrow \underset{\displaystyle \dim R(T) \le \dim W}{\overset{\displaystyle \mathrm{rank}(T)}{\shortparallel}} = \dim V - \mathrm{null}(T) = \dim V$$

$$\Rightarrow \dim V \le \dim W.$$

Continuation of works:

Scrap paper