!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : ID :

1(15pts) For each of the following subsets of $\mathbf{R}^3$, determine whether it is a vector subspace of $\mathbf{R}^3$. Explain your reason.

(1) $\{(a, b, c); a + 99b = 101c\} =: S_1$
(2) $\{(a, b, c); a^2 - b^2 = 0\}. =: S_2$
(3) $\{(a, b, c); a^2 + b^2 = 0\}. =: S_3$

(1) Yes. $S_1 = N((1\ 99\ -101))$
$= \{(a,b,c) : (1\ 99\ -101)\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0\}$. 5

(2) No. $S_2 = \{(a+b)(a-b)=0\} = \{a+b=0\} \cup \{a-b=0\}$. 5

Not closed under addition e.g. $\underset{\in S_2}{(1,-1)} + \underset{\in S_2}{(1,1)} = (2,0) \notin S_2$.

(3) Yes. $S_3 = \{a=b=0\} = \{(0,0,c) : c \in \mathbb{R}\} = \text{Span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right\}$. 5

3(25pts) Consider the following subset S of $P_3(\mathbf{R})$.

$$S = \{1 - x, x + x^2, x^2 + x^3, 1 + x^3\}$$

(1) Is S linearly dependent or linearly independent?
(2) Is $x + x^2 + 2x^3$ contained in $\mathrm{Span}(S)$? Explain your reason.

(1) + (2) Choose standard basis $\beta = \{1, x, x^2, x^3\}$ for $P_3(\mathbb{R})$. Consider: 5

$$A = \left([1-x]_\beta \ [x+x^2]_\beta \ [x^2+x^3]_\beta \ [1+x^3]_\beta \,\middle|\, [x+x^2+2x^3]_\beta \right)$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 2 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 2 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right). \quad 5$$

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$\Rightarrow$ S is linearly independent 5 $\left(\overset{\#S = 4 = \dim P_3(\mathbb{R})}{\Longrightarrow} \ S \text{ is a basis for } P_3(\mathbb{R}) \right)$

and $x + x^2 + 2x^3$ is contained in $\mathrm{Span}(S) = P_3(\mathbb{R})$ 5

Check: $-1 \cdot (1-x) + 0 \cdot (x + x^2) + 1 \cdot (x^2 + x^3) + 1 \cdot (1 + x^3)$

$= (-1+1) + x + 1 \cdot x^2 + (1+1)x^3 = x + x^2 + 2x^3$ ✓

3(20pts) Let r_1, r_2, r_3 be row vectors in $\mathbf{R}^3$. Find the value of k that satisfies the following equality:

$$\det\begin{pmatrix} 3r_1 + r_2 \\ r_1 + r_3 \\ r_1 + 2r_2 \end{pmatrix} = k \cdot \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$$\det\begin{pmatrix} 3r_1 + r_2 \\ r_1 + r_3 \\ r_1 + 2r_2 \end{pmatrix} = \det\begin{pmatrix} 3r_1 \\ r_1 + r_3 \\ r_1 + 2r_2 \end{pmatrix} + \det\begin{pmatrix} r_2 \\ r_1 + r_3 \\ r_1 + 2r_2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 3r_1 \\ r_3 \\ 2r_2 \end{pmatrix} + \det\begin{pmatrix} r_2 \\ r_1 + r_3 \\ r_1 \end{pmatrix}$$

$$= 6 \cdot \det\begin{pmatrix} r_1 \\ r_3 \\ r_2 \end{pmatrix} + \det\begin{pmatrix} r_2 \\ r_3 \\ r_1 \end{pmatrix}$$

$$= -6 \cdot \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - \det\begin{pmatrix} r_2 \\ r_1 \\ r_3 \end{pmatrix}$$

$$= -6 \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = -5 \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$\Rightarrow k = -5.$

4(25pts) Let A be a real square matrix. Assume that v_1 and v_2 are two eigenvectors associated with two different eigenvalues $\lambda_1 \neq \lambda_2$. Prove that v_1, v_2 are linearly independent. What else can you say if moreover A is symmetric?

Proof: $Av_1 = \lambda_1 v_1$, $Av_2 = \lambda_2 v_2$, $\lambda_1 \neq \lambda_2$, $v_1 \neq 0$, $v_2 \neq 0$.

Assume $a_1 v_1 + a_2 v_2 = 0$. Then

$$0 = A(a_1 v_1 + a_2 v_2) = a_1 A(v_1) + a_2 \cdot A(v_2) = a_1 \cdot \lambda_1 v_1 + a_2 \cdot \lambda_2 v_2$$

So: $\begin{cases} a_1 v_1 + a_2 v_2 = 0 \\ a_1 \lambda_1 v_1 + a_2 \cdot \lambda_2 v_2 = 0 \end{cases} \Rightarrow \begin{cases} a_1 \lambda_1 v_1 + a_2 \cdot \lambda_1 v_2 = 0 \\ a_1 \cdot \lambda_1 v_1 + a_2 \lambda_2 v_2 = 0 \end{cases}$

$$\Rightarrow \underbrace{(a_2\lambda_2 - a_2\lambda_1) v_2}_{= a_2(\lambda_2 - \lambda_1) v_2} = 0 \xRightarrow{\lambda_1 \neq \lambda_2} a_2 \cdot v_2 = 0 \xRightarrow{v_2 \neq 0} a_2 = 0$$

$$\Rightarrow a_1 \cdot v_1 = 0 \xRightarrow{v_1 \neq 0} a_1 = 0.$$

So $\{v_1, v_2\}$ is linearly independent.

If A is symmetric, then v_1 is orthogonal to v_2:

证: $\langle Av_1, v_2 \rangle = (Av_1)^T \cdot v_2 = v_1^T \cdot A^T v_2 \xlongequal{A^T = A} v_1^T A v_2$

$\langle Av_1, v_2 \rangle = \langle \lambda_1 v_1, v_2 \rangle = \lambda_1 \langle v_1, v_2 \rangle$, $\quad v_1^T A v_2 = v_1^T(\lambda_2 v_2) = \lambda_2 \langle v_1, v_2 \rangle$

$$\Rightarrow (\lambda_2 - \lambda_1)\langle v_1, v_2 \rangle = 0 \xRightarrow{\lambda_1 \neq \lambda_2} \langle v_1, v_2 \rangle = 0.$$

5(25pts) (1) Use the Gram-Schmidt process to $\{1, x, x^2\}$ to find an orthonomal basis β of $P_2(\mathbf{R})$ with respect to the inner product $\langle f, g\rangle = \int_{-2}^{2} f(x)g(x)dx$.
(2) Find the Fourier coefficients of $h(x) = x^2$ with respect to β.

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(1) $V_1 = W_1 = 1$, $\|W_1\|^2 = \int_{-2}^{2} 1 \cdot 1\, dx = 4 = \|V_1\|^2$

$W_2 = x$, $V_2 = W_2 - \frac{\langle W_2, W_1\rangle}{\|V_1\|^2} V_1$ $\quad \langle W_2, V_1\rangle = \int_{-2}^{2} x \cdot 1\, dx = \frac{x^2}{2}\Big|_{-2}^{2} = 0.$

$\Rightarrow V_2 = x - \frac{0}{4} V_1 = x.$ $\quad \|V_2\|^2 = \int_{-2}^{2} x \cdot x\, dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{16}{3}.$ 5

$V_3 = W_3 - \frac{\langle W_3, V_1\rangle}{\|V_1\|^2} V_1 - \frac{\langle W_3, V_2\rangle}{\|V_2\|^2} V_2$

$\langle W_3, V_1\rangle = \int_{-2}^{2} x^2 \cdot 1\, dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{16}{3}$, $\quad \langle W_3, V_2\rangle = \int_{-2}^{2} x^2 \cdot x\, dx = \frac{x^4}{4}\Big|_{-2}^{2} = 0.$

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$\Rightarrow V_3 = x^2 - \frac{\frac{16}{3}}{4} 1 = x^2 - \frac{4}{3}$

$\|V_3\|^2 = \int_{-2}^{2} (x^2 - \frac{4}{3})^2 dx \underset{\text{even fct.}}{=} 2 \int_0^2 (x^4 - \frac{8}{3}x^2 + \frac{16}{9}) dx = 2 \cdot (\frac{x^5}{5} - \frac{8}{9}x^3 + \frac{16}{9}x)\Big|_0^2$

$= 2 \cdot (\frac{32}{5} - \frac{64}{9} + \frac{32}{9}) = 64 \cdot (\frac{1}{5} - \frac{1}{9}) = \frac{256}{45} = \frac{16^2}{3^2} \cdot \frac{1}{5}$

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$\Rightarrow U_1 = \frac{V_1}{\|V_1\|} = \frac{1}{2}$, $\quad U_2 = \frac{V_2}{\|V_2\|} = \frac{\sqrt{3}}{4} x$, $\quad U_3 = \frac{V_3}{\|V_3\|} = \frac{3\sqrt{5}}{16}(x^2 - \frac{4}{3})$
$= \sqrt{5}\,(\frac{3}{16}x^2 - \frac{1}{4})$

form an o.n.b. for $P_2(\mathbb{R})$.

(2) $x^2 = x^2 - \frac{4}{3} + \frac{4}{3}$

$= \frac{16}{3\sqrt{5}} \cdot \frac{3\sqrt{5}}{16}(x^2 - \frac{4}{3}) + \frac{4\cdot 2}{3} \cdot \frac{1}{2}$

$= \frac{16}{3\sqrt{5}} \cdot u_3 + \frac{8}{3} \cdot u_1$

$= \frac{8}{3} u_1 + 0 \cdot u_2 + \frac{16}{3\sqrt{5}} \cdot u_3$

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Fourier coefficients: $\frac{8}{3}$, 0 , $\frac{16}{3\sqrt{5}} = \frac{16}{15}\sqrt{5}$

$\frac{8}{3} = \langle h, u_1\rangle$, $0 = \langle h, u_2\rangle$, $\frac{16}{3\sqrt{5}} = \langle h, u_3\rangle$.

OR: $\langle h, u_1\rangle = \frac{\langle h, v_1\rangle}{\|v_1\|} = \frac{16}{3} \cdot \frac{1}{2} = \frac{8}{3}$

$\langle h, v_1\rangle = \int_{-2}^{2} x^2 \cdot 1 \, dx = \frac{x^3}{3}\Big|_{-2}^{2} = \frac{16}{3}$

$\langle h, u_2\rangle = \frac{\langle h, v_2\rangle}{\|v_2\|} = 0$, $\langle h, v_2\rangle = \int_{-2}^{2} x^2 \cdot x \, dx = \frac{x^4}{4}\Big|_{-2}^{2} = 0$.

$\langle h, u_3\rangle = \frac{\langle h, v_3\rangle}{\|v_3\|} = \frac{\frac{256}{45}}{\frac{16}{3} \cdot \frac{1}{\sqrt{5}}} = \frac{16}{15} \cdot \sqrt{5}$

$\langle h, v_3\rangle = \int_{-2}^{2} x^2 \cdot (x^2 - \frac{4}{3}) \, dx = 2 \cdot \left[\frac{x^5}{5} - \frac{4}{9}x^3\right]_0^2 = 64 \cdot \frac{4}{45} = \frac{256}{45}$

7(25pts) Let A be a square matrix with characteristic polynomial equal to $(t-2)^{10}$. Assume that the dot diagram of A is the following:

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & & \end{array}$$

(1) Write down the Jordan canonical form of A.
(2) Calculate $\dim R(A-2I)$ and $\dim R((A-2I)^2)$.

(1) $$J(A) = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & & & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} \end{pmatrix}$$ 9

(2) $\dim N(A-2I) = \#\text{dots in 1st. row} = 4$ 4

$\dim N((A-2I)^2) = \#\text{dots in first 2 rows} = 8$ 4

$\Rightarrow \dim R(A-2I) = 10 - \dim N(A-2I) = 10 - 4 = 6$ 4

$\dim R((A-2I)^2) = 10 - \dim N((A-2I)^2) = 10 - 8 = 2.$ 4

8(25pts) Consider the linear transformation:

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R}), \quad T(f(x)) = f''(x) + f'(x) + f(0).$$

(1) Find all eigenvalues of T and the corresponding dot diagrams.
(2) Find a basis γ of $P_2(\mathbf{R})$ such that $[T]_\gamma$ is a Jordan canonical form.

(2)+(1) $\beta = \{1, x, x^2\}$ standard basis for $P_2(\mathbb{R})$.

$$[T]_\beta = \left([T(1)]_\beta \; [T(x)]_\beta \; [T(x^2)]_\beta \right) = \left([1]_\beta \; [1]_\beta \; [2+2x]_\beta \right)$$

$$= \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = A$$

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$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 2 \\ 0 & -\lambda & 2 \\ 0 & 0 & -\lambda \end{vmatrix} = \lambda^2(1-\lambda) = 0 \Rightarrow \begin{matrix} \lambda = 0, & m_0 = 2 \\ \lambda = 1, & m_1 = 1 \end{matrix}$$

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$$\lambda = 0: \; A - 0 \cdot I = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_0 = \operatorname{Span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

5

$\dim E_0 = 1 \Rightarrow$ dot diagram for $\lambda = 0$:

$\bullet\, v_1$
$\bullet\, v_2$

5

choose $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ solve: $(A - 0I)v_2 = v_1$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{choose } v_2 = \begin{pmatrix} -2 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

5

$$\lambda = 1: \; A - 1 \cdot I = \begin{pmatrix} 0 & 1 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow E_1 = \operatorname{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \bullet\, v_3$$

$\Rightarrow$ Jordan canonical basis $\gamma = \{ \overset{f_1}{-1+x}, \overset{f_2}{-2+\frac{1}{2}x^2}, \overset{f_3}{1} \}$ satisfies

$$[T]_\gamma = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Check $T(-1+x) = 0 + 1 + (-1) = 0 \cdot f_1$
$T(-2+\frac{1}{2}x^2) = 1 + x - 2 = x - 1 = 1 \cdot f_1 + 0 \cdot f_2$ ✓
$T(1) = 1 = 1 \cdot f_3$

9(20pts) Find the linear function $f(t) = c_0 + c_1 t$ that has the best fit to the data:

$$\{(0,0),(1,2),(2,1),(3,-1),(4,0)\} = \{(t_i, y_i); 1 \leq i \leq 5\}.$$

with respect to the error: $\mathbf{E} = \sum_{i=1}^{5}(c_0 + c_1 t_i - y_i)^2$.

Set $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \qquad y = \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \\ 0 \end{pmatrix}$

$$A^T A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 10 & \underset{\substack{\uparrow \\ 1+4+9+16}}{30} \end{pmatrix} = 5 \cdot \begin{pmatrix} 1 & 2 \\ 2 & 6 \end{pmatrix}.$$

$$(A^T A)^{-1} = \frac{1}{5} \cdot \frac{1}{6-4} \begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{10} \cdot \begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix}.$$

$$A^T y = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 6 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 10 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ -\frac{3}{10} \end{pmatrix}$$

$\Rightarrow$ $f(t) = 1 - \frac{3}{10} t$ has the best fit to the data.

8(20pts) Find the linear function $f(t) = c_0 + c_1 t$ that has the best fit to the data:

$$\{(0,0),(1,2),(2,1),(3,-1),(4,0)\} = \{(t_i, y_i); 1 \le i \le 5\}.$$

with respect to the error: $\mathbf{E} = \sum_{i=1}^{5}(c_0 + c_1 t_i - y_i)^2$.

$$E = \left\| \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \\ 1 & t_5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} - \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{pmatrix} \right\|^2 = \|A \cdot v - y\|^2.$$

Find orthogonal projection of y to column space of A

$$\text{column space of } A = \operatorname{Span}\left\{ \begin{pmatrix} 1\\1\\1\\1\\1 \end{pmatrix}, \begin{pmatrix} 0\\1\\2\\3\\4 \end{pmatrix} \right\}, \quad w_1 = \begin{pmatrix} 1\\1\\1\\1\\1 \end{pmatrix},\ w_2 = \begin{pmatrix} 0\\1\\2\\3\\4 \end{pmatrix}$$

Use Gram–Schmidt:

$v_1 = w_1$, $\|v_1\|^2 = 5$, $\langle w_2, v_1 \rangle = 10$.

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 0\\1\\2\\3\\4 \end{pmatrix} - \frac{10}{5}\begin{pmatrix} 1\\1\\1\\1\\1 \end{pmatrix} = \begin{pmatrix} -2\\-1\\0\\1\\2 \end{pmatrix}. \quad \|v_2\|^2 = 4+1+1+4 = 10$$

$$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt5}\begin{pmatrix} 1\\1\\1\\1\\1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{10}}\begin{pmatrix} -2\\-1\\0\\1\\2 \end{pmatrix}, \quad y = \begin{pmatrix} 0\\2\\1\\-1\\0 \end{pmatrix}.$$

$$\langle y, u_1\rangle = \frac{1}{\sqrt5}2, \quad \langle y, u_2\rangle = \frac{1}{\sqrt{10}}\cdot(0-2+0-1+0) = -\frac{3}{\sqrt{10}}$$

Orthogonal Projection:
$$\langle y, u_1\rangle u_1 + \langle y, u_2\rangle u_2 = \frac{1}{5}\cdot 2\cdot\begin{pmatrix} 1\\1\\1\\1\\1 \end{pmatrix} - \frac{3}{10}\cdot\begin{pmatrix} -2\\-1\\0\\1\\2 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 10\\7\\4\\1\\-2 \end{pmatrix} = y_1$$

Solve $A \cdot v = y_1$

$$\left(\begin{array}{cc|c} 1 & 0 & 10 \\ 1 & 1 & 7 \\ 1 & 2 & 4 \\ 1 & 3 & 1 \\ 1 & 4 & -2 \end{array}\right) = \left(\begin{array}{cc|c} 1 & 0 & 10 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \\ 0 & 3 & -9 \\ 0 & 4 & -12 \end{array}\right) \Rightarrow v = \frac{1}{10}\begin{pmatrix} 10 \\ -3 \end{pmatrix} \Rightarrow y = \frac{1}{10}(10 - 3t) \text{ is the best linear approximation.}$$

9(20pts) Let $S : V \to W$ and $T : W \to V$ be linear transformations between two vector spaces V and W. Let $T \circ S : V \to V$ be the composition.

(1) Prove that if S is surjective (i.e onto), then $\text{Rank}(T \circ S) = \text{Rank}(T)$.

(2) If S is injective (i.e. one-to-one), is $\text{Rank}(T \circ S) = \text{Rank}(T)$ always true? Prove this statement or find a counterexample.

(1) Proof: $\text{Rank}(T\circ S) = \dim R(T\circ S)$.

If S is onto, then $R(T\circ S) = \{T\circ S(v) = T(S(v)) : v \in V\}$
$= \{T(w) : w \in W\} = R(T)$

$\Rightarrow \dim R(T\circ S) = \dim R(T) = \text{Rank}(T)$.

(2) Not always true.

Example: $S: \mathbb{R} \to \mathbb{R}^2$, $x \mapsto (x, 0)$, $T = Id: \mathbb{R}^2 \to \mathbb{R}^2$, $(x,y) \mapsto (x,y)$

$\text{Rank}(T\circ S) = 1 < \text{Rank}(T) = 2$.