!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(9pts) For each of the following subsets of $\mathbf{R}^3$, determine whether it is a vector subspace of $\mathbf{R}^3$. Explain your reason.

(1) $\{(a, b, c); a^2 + b = 0\}$. $= S_1$
(2) $\{(a, b, c); a + b + c = 1\}$. $= S_2$
(3) $\{(a, b, c); c = 2a - b\}$. $= S_3$

(1) Not a subspace: Not closed under addition:

$(-1,-1)$, $(1,-1) \in S_1$. $(-1,-1) + (1,-1) = (0,-2) \notin S_1$.

(2) Not a subspace: $(0,0,0) \notin S_2$

(3) $S_3 = \{ 2a - b - c = 0 \} = N\begin{pmatrix} 2 & -1 & -1 \end{pmatrix}$ null space is a subspace.

2(12pts) Consider the following subset S of $P_3(\mathbf{R})$.

$$S = \{1 - 2x, x + x^2 - x^3, 1 + 2x^2 - 2x^3, 1 + x^3\}.$$

(1) Is S linearly dependent or linearly independent?
(2) Is $f(x) = x$ contained in Span(S)? Explain your reason.

(1) + (2) Use coordinates w.r.t. Standard basis $\beta = \{1, x, x^2, x^3\}$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ -2 & 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & -1 & -2 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\| \qquad\qquad \downarrow$$

$$(A \mid b) \qquad\qquad \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

3

3

$\Rightarrow$ $\text{rank}(A) = 3 < 4 = \text{rank}(A \mid b)$

$\Rightarrow$ S is linearly dependent and

3

$Av = b$ has no solution $\Longrightarrow$ $f(x) = x$ is not contained in Span(S).

3

3(10pts) Let r_1, r_2, r_3 be row vectors in $\mathbf{R}^3$. Find the value of k that satisfies the following equality:

$$\det\begin{pmatrix} r_2+3r_3 \\ r_1-r_3 \\ 2r_1+r_2 \end{pmatrix} = k\cdot\det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$$\det\begin{pmatrix} r_2+3r_3 \\ r_1-r_3 \\ 2r_1+r_2 \end{pmatrix} = \det\begin{pmatrix} r_2 \\ r_1-r_3 \\ 2r_1+r_2 \end{pmatrix} + \det\begin{pmatrix} 3r_3 \\ r_1-r_3 \\ 2r_1+r_2 \end{pmatrix}$$ 4

$$= \det\begin{pmatrix} r_2 \\ r_1-r_3 \\ 2r_1 \end{pmatrix} + \det\begin{pmatrix} 3r_3 \\ r_1 \\ 2r_1+r_2 \end{pmatrix}$$

$$= \det\begin{pmatrix} r_2 \\ -r_3 \\ 2r_1 \end{pmatrix} + \det\begin{pmatrix} 3r_3 \\ r_1 \\ r_2 \end{pmatrix}$$ 4

$$= -\det\begin{pmatrix} 2r_1 \\ -r_3 \\ r_2 \end{pmatrix} - \det\begin{pmatrix} r_1 \\ 3r_3 \\ r_2 \end{pmatrix}$$

$$= \det\begin{pmatrix} 2r_1 \\ r_2 \\ -r_3 \end{pmatrix} + \det\begin{pmatrix} r_1 \\ r_2 \\ 3r_3 \end{pmatrix}$$

$$= -2\cdot\det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + 3\cdot\det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} = \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}$$

$\Rightarrow k=1.$ 2

4(10pts) Let A be a square matrix, and $\{v_1, v_2\}$ be two non-zero vectors satisfying:

$$Av_1 = \lambda v_1, \quad Av_2 = v_1 + \lambda v_2.$$

In other words, $\{v_1, v_2\}$ is a cycle of generalized eigenvectors associated to λ. Is $\{v_1, v_2\}$ always linearly independent? Prove your statement or find a counterexample.

$\{v_1, v_2\}$ is always linearly independent. 4

Proof: Assume $a_1 v_1 + a_2 v_2 = 0$.

Apply $(A - \lambda I)$: $a_1 \underbrace{(A-\lambda I) v_1}_{=0} + a_2 \underbrace{(A-\lambda I) v_2}_{=a_2 v_1} = 0$ 4

$\Rightarrow a_2 v_1 = 0 \xRightarrow{v_1 \neq 0} a_2 = 0$ 2

$\Rightarrow a_1 v_1 = -a_2 v_2 = 0 \xRightarrow{v_1 \neq 0} a_1 = 0$.

So $\{v_1, v_2\}$ is linearly independent.

5(12pts) Find an orthogonal matrix S such that S^TAS is diagonal where

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = \begin{vmatrix} 0 & \lambda+1 & -\lambda^2+1 \\ 0 & -\lambda-1 & 1+\lambda \\ 1 & 1 & -\lambda \end{vmatrix} = 1\cdot(\lambda+1)\begin{vmatrix} \lambda+1 & -(\lambda+1)(\lambda-1) \\ -1 & 1 \end{vmatrix}$$

$$= 1\cdot(\lambda+1)^2\cdot\begin{vmatrix} 1 & -(\lambda-1) \\ -1 & 1 \end{vmatrix} = (\lambda+1)^2\cdot(1-(\lambda-1)) = (\lambda+1)^2(2-\lambda) \qquad 3$$

$\Rightarrow$ eigenvalues: $\lambda=-1$, $\text{mult}(-1)=2$; $\lambda=2$, $\text{mult}(2)=1$.

$$\lambda=-1, \quad A-(-1)I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A+I) = \text{Span}\left\{ \overset{w_1}{\overset{\parallel}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}}, \overset{w_2}{\overset{\parallel}{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}} \right\}$$

$$\lambda=2: \quad A-2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & -3 & 3 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad 3$$

$$\Rightarrow N(A-2I) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$$

- Gram-Schmidt to $\{w_1, w_2, w_3\}$. Note $w_3 \perp \{w_1, w_2\}$

$v_1 = w_1$, $\|v_1\|^2 = 2$, $\langle w_2, v_1\rangle = 1$.

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}. \qquad v_3 = w_3$$

$$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}, \quad u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad 6$$

$$\Rightarrow S = (u_1\ u_2\ u_3) = \begin{pmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{pmatrix} \text{ satisfies } S^TAS = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

6(12pts) Let A be a square matrix with characteristic polynomial equal to $(t-2)^{10}$. Assume that the dot diagram of A is the following:

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \\ \bullet & & & & \end{array}$$

(1) Write down the Jordan canonical form of A.
(2) Calculate $\dim N((A-2I)^2)$ and $\dim R(A-2I)$.

(1) $$J(A) = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & & & & \\ & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & & & \\ & & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & & \\ & & & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & & & & \boxed{2} \end{pmatrix}$$ 6

(2) $\dim N(A-2I) =$ # dots of 1st. row $= 5$ 2

$\dim N((A-2I)^2) =$ # dots of 1st. & 2nd. row $= 9$. 2

$\dim R(A-2I) = 10 - \dim N(A-2I) = 10 - 5 = 5$. 2

7(12pts) Consider the linear transformation:

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R}), \quad T(f(x)) = f''(x) + f(1)x.$$

(1) Find all eigenvalues of T and the corresponding dot diagrams.
(2) Find a basis γ of $P_2(\mathbf{R})$ such that $[T]_\gamma$ is a Jordan canonical form.

(1) $\beta = \{1, x, x^2\}$, $T(1) = 0 + 1\cdot x = x$, $T(x) = 0 + 1\cdot x = x$, $T(x^2) = 2 + 1\cdot x = 2 + x$

$$[T]_\beta = \left([T(1)]_\beta \; [T(x)]_\beta \; [T(x^2)]_\beta\right) = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} = A.$$ (2)

$$A - \lambda I = \begin{vmatrix} -\lambda & 0 & 2 \\ 1 & 1-\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} = (-\lambda)\cdot \begin{vmatrix} -\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (-\lambda)(-\lambda)(1-\lambda) = \lambda^2(1-\lambda)$$

$\Rightarrow \lambda = 0$. $\text{mult}(0) = 2$; $\lambda = 1$, $\text{mult}(1) = 1$. (2)

$\lambda = 0$: $A - 0\cdot I = \begin{pmatrix} 0 & 0 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_0 = \text{Span}\Big\{\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}_{v_1}\Big\}$. $\dim E_0 = 1 < \underset{2}{\text{mult}(0)}$ (2)

$\Rightarrow$ dot diagram for $\lambda = 0$: $\bullet\, v_1$ $\bullet\, v_2$. (2) dot diagram for $\lambda = 1$: $\bullet\, v_3$ (1)

(2) Solve $(A - 0I)v_2 = v_1$: $\left(\begin{array}{ccc|c} 0 & 0 & 2 & -1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow v_2 = \begin{pmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \end{pmatrix}$ (2) verify: $\frac{3}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ ✓

$\lambda = 1$: $A - 1\cdot I = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_1 = \text{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right\}$. (1)

$\Rightarrow \gamma = \left\{-1 + x,\ \frac{3}{2} - \frac{1}{2}x^2,\ x\right\}$ is a Jordan canonical basis.

Verify:

$T(-1+x) = 0 + 0\cdot x$, $T\left(\frac{3}{2} - \frac{1}{2}x^2\right) = -1 + 1\cdot x$, $T(x) = 0 + 1\cdot x = x$ ✓

8(11pts) Find the linear function $f(t) = c_0 + c_1 t$ that has the best fit to the data:

$$\{(1,2),(2,1),(3,-1)\} = \{(t_i, y_i); 1 \le i \le 3\}.$$

with respect to the error: $\mathbf{E} = \sum_{i=1}^{3}(c_0 + c_1 t_i - y_i)^2$.

$$E = \left\| \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\|^2 = \| A\cdot v - y \|^2. \quad A = (w_1 \ w_2) = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}.$$

2

Gram-Schmidt for $\{w_1, w_2\}$: $v_1 = w_1$, $\|v_1\|^2 = 3$. $\langle w_2, v_1 \rangle = 6$.

$$v_2 = w_2 - \frac{\langle w_2, v_1 \rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \frac{6}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

3

$$\Rightarrow \quad u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

$\langle y, u_1 \rangle = \frac{1}{\sqrt{3}} 2$

$\langle y, u_2 \rangle = \frac{1}{\sqrt{2}}(-2-1) = -\frac{3}{\sqrt{2}}$.

$$\text{Proj } y = \langle y, u_1 \rangle u_1 + \langle y, u_2 \rangle u_2$$

3

$$= \frac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \frac{3}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 13 \\ 4 \\ -5 \end{pmatrix} = z$$

3

Solve $Av = z$:

$$\left(\begin{array}{cc|c} 1 & 1 & 13 \\ 1 & 2 & 4 \\ 1 & 3 & -5 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1 & 13 \\ 0 & 1 & -9 \\ 0 & 2 & -18 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 22 \\ 0 & 1 & -9 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow v = \begin{pmatrix} 22 \\ -9 \end{pmatrix}$$

$\Rightarrow c_0 + c_1 \cdot t = (22 - 9t)\frac{1}{6} = \frac{11}{3} - \frac{3}{2}t$ is the best fit.

OR. $$A^T A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ 6 & 14 \end{pmatrix}. \quad \det(A^T A) = 42 - 36 = 6$$

2

$$\begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = (A^T A)^{-1} A^T y = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 14 & -6 \\ -6 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 22 \\ -9 \end{pmatrix}$$

6

3 ✓

9(12pts) Let $S : V \to W$ and $T : W \to V$ be linear transformations between two vector spaces V and W. Let $T \circ S : V \to V$ be the composition.

(1) If T is injective (i.e. one-to-one), is $\text{Rank}(T \circ S) = \text{Rank}(S)$ always true? Prove your statement or find a counterexample.

(2) If T is surjective (i.e. onto), is $\text{Rank}(T \circ S) = \text{Rank}(S)$ always true? Prove your statement or find a counterexample.

(1) True. 3 $R(T\circ S) = \{T\circ S(v) : v \in V\} = \{T(w) : w \in R(S)\}$

$T|_{R(S)} : R(S) \to V$ is one-to-one $\Rightarrow N(T|_{R(S)}) = \{0\}$

$\Rightarrow \underbrace{\dim(T(R(S)))}_{\text{rank}(T\circ S)} = \dim R(S) - \dim N(T|_{R(S)}) = \underbrace{\dim R(S)}_{\text{rank}(S)}$. 3

(2) Not true. 3 counter-example: $S : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x$

$T : \mathbb{R} \to \mathbb{R} \quad x \mapsto 0$ 3

$\underbrace{\text{Rank}(T\circ S)}_{0} \neq \underbrace{\text{Rank}(S)}_{1}$.