# MATH 350 FALL 2025 MIDTERM II

**NAME:** **ID:**

THERE ARE FOUR (4) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

| | | |
|---|---|---|
| 1 | | 24pts |
| 2 | | 26pts |
| 3 | | 26pts |
| 4 | | 24pts |
| Total | | 100pts |

**!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!**

**NAME :** **ID :**

**1(24pts)** Let $A$ be a $3 \times 5$ matrix such that an elementary row operation changes $A$ to become (Note: not a reduced row ethelon form):

$$\begin{pmatrix} 1 & 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} = B$$

(1) Find a basis for the null space $N(A)$.

(2) Assume that $A = (v_1\ v_2\ v_3\ v_4\ v_5)$ where $v_i$ denotes the $i$-th column satisfies

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Recover the matrix $A$.

(1) $B = EA$ with $E$ invertible $\Rightarrow N(A) = N(B)$.

$$B \xrightarrow{-③+①} \begin{pmatrix} 1 & 1 & 0 & 0 & -3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \xrightarrow{-②+①} \begin{pmatrix} 1 & 0 & -1 & 0 & -4 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \Leftrightarrow \begin{cases} x_1 = x_3 + 4x_5 \\ x_2 = -x_3 - x_5 \\ x_4 = -2x_5 \end{cases}$$ 6

$$\Rightarrow N(B) = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\} = N(A).$$ 6

Vectors in $N(A)$ gives linear relation between column vectors of $A$:

(2) $$+v_1 - v_2 + v_3 = 0, \quad 4v_1 - v_2 - 2v_4 + v_5 = 0.$$

Check: $$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 4 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} + \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \checkmark$$

$$\Rightarrow v_3 = -v_1 + v_2 = -\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$ 6

$$v_5 = -4v_1 + v_2 + 2v_4 = -\begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 2 \end{pmatrix}.$$ 6

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & -1 & 0 & -4 \\ 1 & 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

**2(26pts)** Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be the linear transformation given by

$$T(f) = f''(x) + xf'(x) + f(0).$$

Let $\beta = \{1, x, x^2\}$ be the standard basis.

(1) Calculate the matrix representation $A = [T]_\beta$ relative to the standard basis $\beta$.

(2) Determine whether $T$ is diagonalizable or not. If yes, then find $Q$ such that $Q^{-1}AQ$ is diagonal.

(1) $$[T]_\beta = \left[\ \underset{\underset{\begin{pmatrix}1\\0\\0\end{pmatrix}}{\|}}{\underset{[1]_\beta}{\underset{\|}{[T(1)]_\beta}}} \quad \underset{\underset{\begin{pmatrix}0\\1\\0\end{pmatrix}}{\|}}{\underset{[x\cdot 1+0]_\beta}{\underset{\|}{[T(x)]_\beta}}} \quad \underset{\underset{\begin{pmatrix}2\\0\\2\end{pmatrix}}{\|}}{\underset{[2+2x^2]_\beta}{\underset{\|}{[T(x^2)]_\beta}}}\ \right] = \begin{pmatrix}1&0&2\\0&1&0\\0&0&2\end{pmatrix} = A$$

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(2) $$|A-\lambda I| = \begin{vmatrix}1-\lambda & 0 & 2\\ 0 & 1-\lambda & 0\\ 0 & 0 & 2-\lambda\end{vmatrix} = (1-\lambda)^2(2-\lambda) = 0 \Rightarrow \begin{array}{l}\lambda = 1,\ \text{mult}(1) = 2 \quad 4\\ \lambda = 2,\ \text{mult}(2) = 1. \quad 4\end{array}$$

$\lambda=1$: $$A - I = \begin{pmatrix}0&0&2\\0&0&0\\0&0&1\end{pmatrix} \to \begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix} \Rightarrow \underset{\underset{E_1}{\|}}{N(A-I)} = \text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\} \quad 6$$

$\dim E_1 = 2 = \text{mult}(1)$, $\dim E_2 = 1 = \text{mult}(1) \Rightarrow A$ is diagonalizable.

$\lambda=2$: $$A - 2I = \begin{pmatrix}-1&0&2\\0&-1&0\\0&0&0\end{pmatrix} \to \begin{pmatrix}1&0&-2\\0&1&0\\0&0&0\end{pmatrix} \Rightarrow N(A-2I) = \text{Span}\left\{\begin{pmatrix}2\\0\\1\end{pmatrix}\right\} \quad 3$$

$$\Rightarrow Q = \begin{pmatrix}1&0&2\\0&1&0\\0&0&1\end{pmatrix} \text{ satisfies } Q^{-1}\cdot A\cdot Q = \begin{pmatrix}1&0&0\\0&1&0\\0&0&2\end{pmatrix}.$$

Check: $T(1) = 1$, $T(x) = x$, $T(2+x^2) = 2 + x\cdot 2x + 2 = \underset{\underset{2(2+x^2)}{\|}}{4+2x^2}$ ✓

**3(26pts)** Let $V$ be the subspace of $P_3(\mathbf{R}) = \mathrm{Span}\{1, x, x^2, x^3\}$ spanned by

$$\underbrace{1 + x + x^2}_{v_1}, \quad \underbrace{2 + x + x^3}_{v_2}, \quad \underbrace{x + 2x^2 - x^3}_{v_3}, \quad \underbrace{1 - x^2 + x^3}_{v_4}$$

(1) Find a basis $\beta$ for $V$.

(2) Extend $\beta$ to a basis $\gamma$ of $P_3(\mathbf{R})$. Explain why the subset $\gamma$ you obtain is a basis for $P_3(\mathbf{R})$.

(1) $$\left([v_1]_\beta \ [v_2]_\beta \ [v_3]_\beta \ [v_4]_\beta\right) = \begin{pmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & -2 & 2 & -2 \\ 0 & 1 & -1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \underbrace{\{v_1, v_2\}}_{\beta}$ corresponding to leading variables form a basis for $V$.

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(2) $\{1+x+x^2,\ 2+x+x^3,\ 1,\ x\}$ is a basis of $P_3(\mathbb{R})$

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because it spans $P_3(\mathbb{R})$:

$x^2 = (1+x+x^2) - 1 - x.$

$x^3 = (2+x+x^3) - 2\cdot 1 - x$

**4(24pts)** Assume that $V$ is a vector space with a basis $\beta = \{v_1, v_2, v_3, v_4\}$. Let $T : V \to V$ be a linear transformation that satisfies:

$$Tv_1 = v_1, \quad Tv_2 = v_1 + v_2, \quad Tv_3 = -v_3, \quad Tv_4 = v_3 - v_4.$$

(1) The linear transformation $T$ satisfies the identity:

$$T^4 = a \cdot T^2 + b \cdot T + c \cdot \mathrm{Id}_V$$

where $\mathrm{Id}_V$ is the identity transformation of $V$. Use Cayley-Hamilton theorem to find the numbers $a$, $b$ and $c$.

(2) Calculate the matrix representation of $T^{10}$ relative to $\beta$: $[T^{10}]_\beta$.

(1). $[T]_\beta = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} = A = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$ 4

$|A - \lambda I| = (1-\lambda)^2(-1-\lambda)^2 = (\lambda^2-1)^2 = \lambda^4 - 2\lambda^2 + 1$ 4

By Cayley-Hamilton Thm, $T^4 - 2T^2 + \mathrm{Id}_V = 0 \Rightarrow a = 2, b = 0, c = -1$.

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(2) $A^{10} = \begin{bmatrix} J_1^{10} & 0 \\ 0 & J_2^{10} \end{bmatrix}$. Let $J = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

$J^{10} = \left(\lambda I + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)^{10} = \lambda^{10} \cdot I^{10} + 10 \cdot \lambda^9 \cdot I^9 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0$

$= \begin{pmatrix} \lambda^{10} & 10\lambda^9 \\ 0 & \lambda^{10} \end{pmatrix}$ 6

$\Rightarrow A^{10} = \begin{pmatrix} \boxed{\begin{matrix} 1 & -10 \\ 0 & 1 \end{matrix}} & \\ & \boxed{\begin{matrix} 1 & 10 \\ 0 & 1 \end{matrix}} \end{pmatrix} = [T^{10}]_\beta$ 6

Continuation of work:

Scrap paper