HW5:

2.3.3 $\quad g(x) = 3+x,\quad T: P_2(\mathbb{R}) \to P_2(\mathbb{R}),\quad U: P_2(\mathbb{R}) \to \mathbb{R}^3$

$T(f(x)) = f'(x)\cdot g(x) + 2f(x),\quad U(a+bx+cx^2) = (a+b, c, a-b).$

$\beta = \{1, x, x^2\},\quad \gamma = \{e_1, e_2, e_3\}$

(a) $\cdot\ [U]_\beta^\gamma = \left([U(1)]_\gamma \ \ [U(x)]_\gamma \ \ [U(x^2)]_\gamma \right) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$

$[U(1)]_\gamma = [(1,0,1)]_\gamma,\quad [U(x)]_\gamma = [(1,0,-1)]_\gamma,\quad [U(x^2)]_\gamma = [(0,1,0)]_\gamma$

$\cdot\ T(1) = 1'\cdot(3+x) + 2\cdot 1 = 2$

$T(x) = x'\cdot(3+x) + 2\cdot x = 3+3x \quad \Rightarrow [T]_\beta = \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix}$

$T(x^2) = 2x\cdot(3+x) + 2x^2 = 6x + 4x^2$

$\cdot\ UT(1) = U(2) = (2, 0, 2)$

$UT(x) = U(3+3x) = (6, 0, 0) \quad \Rightarrow [UT]_\beta^\gamma = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix}$

$UT(x^2) = U(6x+4x^2) = (6, 4, -6)$

Use Thm 2.11 to verify:

$$[U]_\beta^\gamma \cdot [T]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{pmatrix} \quad \checkmark$$

(b) $h(x) = 3 - 2x + x^2 \Rightarrow [h(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$U(h(x)) = (1, 1, 5) \Rightarrow [U(h(x))]_\gamma = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix}$

Use Thm 2.14 to verify:

$$[U]_\beta^\gamma \cdot [h(x)]_\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 5 \end{pmatrix} \quad \checkmark$$

4 (d) $T: P_2(\mathbb{R}) \to \mathbb{R}$ $T(f(x)) = f(2)$. $\beta = \{1, x, x^2\}$, $\gamma = \{1\}$

$$T(a_0 + a_1 x + a_2 x^2) = a_0 + 2a_1 + 4a_2 = (1 \ 2 \ 4)\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

$$\Rightarrow [T]_\beta^\gamma = (1 \ 2 \ 4)$$

$$f(x) = 6 - x + 2x^2 \Rightarrow [f]_\beta = \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$$

$$[T(f(x))]_\gamma = [T]_\beta^\gamma \cdot [f]_\beta = (1 \ 2 \ 4)\begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix} = 6 - 2 + 8 = 12$$

12. $T: V \to W$, $U: W \to V$ linear transformations

(a) Prove UT one-to-one $\Rightarrow T$ is one-to-one

Proof: Let $v_1, v_2 \in V$.

$$T(v_1) = T(v_2) \Rightarrow UT(v_1) = UT(v_2) \xRightarrow{UT \text{ one-to-one}} v_1 = v_2.$$ ∎

Must U be one-to-one? No. For example:

$T: \mathbb{R} \to \mathbb{R}^2$ $T(a_1) = (a_1, 0)$ $\Rightarrow UT(a_1) = U(a_1, 0) = a_1$

$U: \mathbb{R}^2 \to \mathbb{R}$ $U(a_1, a_2) = a_1$ but U is not onto.

(b) Prove UT is onto $\Rightarrow U$ is onto.

Proof: Because $UT: V \to V$ is onto, $\forall v \in V$, $\exists v_1 \in V$ s.t. $UT(v_1) = v \Rightarrow w = T(v_1) \in W$ satisfies $U(w) = v$. ∎

Must T be onto? No. The example in (a):

$UT(a_1) = a_1$ is onto, but T is not onto.

(c) Prove that if U and T are one-to-one and onto, then UT is also one-to-one and onto.

Proof: "one-to-one": Let $v_1, v_2 \in V$

$$UT(v_1) = UT(v_2) \xRightarrow{U \text{ one-to-one}} T(v_1) = T(v_2) \xRightarrow{T \text{ one-to-one}} v_1 = v_2 \checkmark$$

"onto" $\forall\ v \in V$, because U is onto, $\exists\ w \in W$ s.t. $U(w) = v$.

Because T is onto, $\exists\ v_1 \in V$ s.t. $T(v_1) = w$.

So $UT(v_1) = U(w) = v$. So $R(UT) = V$ i.e. UT is onto. ∎

2.4 2(f): $T: M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ Determine whether T is invertible.

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}.$$

Because $\dim V = \dim W$, T is invertible $\Longleftrightarrow N(T) = \{0\}$.

Find $N(T)$: $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{cases} a = 0 \\ c = 0 \\ b = 0 \\ d = 0. \end{cases}$

So $N(T) = \{0\}$ and T is invertible.

9. A, B : $n \times n$ matrices s.t. AB is invertible.

(a) Prove that AB are invertible.

<u>Proof</u>: Consider the linear transformations:

$$L_A : \mathbb{R}^n \to \mathbb{R}^n \qquad\qquad L_B : \mathbb{R}^n \to \mathbb{R}^n$$
$$x \mapsto A \cdot x \qquad\qquad\qquad x \mapsto Bx$$

Then $L_{AB} = L_A \cdot L_B$

AB invertible $\Rightarrow L_{AB} = L_A \cdot L_B$ is invertible

$\Rightarrow L_B$ is one-to-one and L_A is onto.

- $L_B : \mathbb{R}^n \to \mathbb{R}^n$ one-to-one $\Rightarrow L_B$ is onto.

  $(\text{nullity}(L_B) = 0 \Rightarrow \text{rank}(L_B) = n - 0 = n \Rightarrow R(L_B) = \mathbb{R}^n)$

  So L_B is invertible $\Rightarrow B$ is invertible.

- $L_A : \mathbb{R}^n \to \mathbb{R}^n$ is onto $\Rightarrow L_A$ is one-to-one

  $(\text{rank}(L_A) = n \Rightarrow \text{nullity}(L_A) = n - n = 0 \Rightarrow N(L_A) = \{0\})$

  So L_A is invertible $\Rightarrow A$ is invertible

(b) Examples of nonsquare matrices: $A = (1 \ \ 0)$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\underline{AB = (1)}$

not invertible

invertible

15. $\dim V = \dim W = n$. $T: V \to W$ linear.

β is a basis for V

Prove: T is an isomorphism iff $T(\beta)$ is a basis for W.

Proof: "if" Let $\beta = \{v_1, \cdots, v_n\}$.

Assume $T(\beta) = \{T(v_1), \cdots, T(v_n)\}$ is a basis for W.

Because $\dim V = \dim W$, to show T is invertible, it is enough to show that T is one-to-one. (By [Thm 2.5, P.71])

$\forall\ v \in N(T)$, write v as the linear combination:

$$v = a_1 \cdot v_1 + \cdots + a_n \cdot v_n$$

Then $0 = T(v) = a_1 T(v_1) + \cdots + a_n T(v_n)$.

$T(\beta)$ is a basis for $W \Rightarrow \{T(v_1), \cdots, T(v_n)\}$ is linearly independent

So $a_1 = \cdots = a_n = 0$, i.e. $v = 0$.

Hence $N(T) = 0 \Rightarrow T$ is one-to-one.

"only if" Assume T is invertible.

To show $T(\beta)$ is a basis, it is enough to show that $T(\beta)$ is linearly independent (by [Corollary 2.(b)], Pg. 48).

Assume $a_1 \cdot T(v_1) + \cdots + a_n \cdot T(v_n) = 0$.

Then $T(a_1 v_1 + \cdots + a_n \cdot v_n) = 0$

Because T is one-to-one, $a_1 \cdot v_1 + \cdots + a_n \cdot v_n = 0$

Because $\{v_1, \cdots, v_n\}$ is linearly independent. $a_1 = \cdots = a_n = 0$

So $\{T(v_1), \cdots, T(v_n)\}$ is indeed linearly independent $\blacksquare$