6.1: 3. $f(t)=t,\ g(t)=e^t \in C([0,1])$

$$\langle f,g\rangle = \int_0^1 f(t)g(t)\,dt = \int_0^1 t\cdot e^t dt = \int_0^1 t\,de^t$$

$$= t\cdot e^t\Big|_0^1 - \int_0^1 e^t dt = e - e^t\Big|_0^1 = e-(e-1) = 1.$$

$$\|f\|^2 = \langle f,f\rangle = \int_0^1 t\cdot t\,dt = \frac{1}{3}t^3\Big|_0^1 = \frac{1}{3} \Rightarrow \|f\| = \frac{1}{\sqrt{3}}$$

$$\|g\|^2 = \langle g,g\rangle = \int_0^1 e^t\cdot e^t dt = \frac{1}{2}e^{2t}\Big|_0^1 = \frac{1}{2}(e^2-1) \Rightarrow \|g\| = \sqrt{\frac{e^2-1}{2}}$$

$$\|f+g\|^2 = \langle f+g, f+g\rangle = \langle f,f\rangle + \langle f,g\rangle + \langle g,f\rangle + \langle g,g\rangle$$

$$= \|f\|^2 + 2\langle f,g\rangle + \|g\|^2 = \frac{1}{3} + 2\cdot 1 + \frac{e^2-1}{2} = \frac{11}{6} + \frac{e^2}{2}$$

$$\Rightarrow \|f+g\| = \left(\frac{11}{6} + \frac{e^2}{2}\right)^{\frac{1}{2}}.$$

Cauchy-Schwarz: $|\langle f,g\rangle|^2 = 1$. $\|f\|^2\cdot\|g\|^2 = \frac{1}{3}\cdot\frac{e^2-1}{2} = \frac{e^2-1}{6}$

$$e^2-1 > 2.7^2-1 = 7.29-1 = 6.29 > 6 \Rightarrow |\langle f,g\rangle|^2 \le \|f\|^2\|g\|^2$$

Triangle inequality: $\frac{1}{\sqrt{3}} \sim 0.577$ $\sqrt{\frac{e^2-1}{2}} \sim 1.787$

$$\left(\frac{11}{6} + \frac{e^2}{2}\right)^{\frac{1}{2}} \sim 2.351 < \underset{\overset{\|}{2.364}}{0.577+1.787}$$

$$\Rightarrow \quad \|f+g\| < \|f\| + \|g\|$$

9. (a) Assume $\langle x, z\rangle = 0 \quad \forall z \in \beta = \{v_1, v_2, \cdots, v_n\}$

β is a basis $\Rightarrow$ $x = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$

$$\Rightarrow \underset{\|x\|^2}{\overset{\shortparallel}{\langle x, x\rangle}} = \langle x, a_1 v_1 + a_2 v_2 + \cdots + a_n v_n\rangle$$

$$= a_1\langle x, v_1\rangle + a_2\langle x, v_2\rangle + \cdots + a_n\langle x, v_n\rangle$$

$$= a_1 \cdot 0 + a_2 \cdot 0 + \cdots + a_n \cdot 0 = 0$$

$\Rightarrow \quad x = 0.$

(b) $\langle x, z\rangle = \langle y, z\rangle \quad \forall z \in \beta$

$\Rightarrow \langle x-y, z\rangle = \langle x, z\rangle - \langle y, z\rangle = 0, \quad \forall z \in \beta$

$\overset{(a)}{\Longrightarrow} x - y = 0$ i.e. $x = y$.

11. $\|x+y\|^2 + \|x-y\|^2$

$= \langle x+y, x+y\rangle + \langle x-y, x-y\rangle$

$= \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle$

$= 2\langle x, x\rangle + 2\langle y, y\rangle = 2\|x\|^2 + 2\|y\|^2.$ (Parallelogram law)

20 (a) $\frac{1}{4}\|x+y\|^2 - \frac{1}{4}\|x-y\|^2$

$= \frac{1}{4}\left(\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle - (\langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle)\right)$

$= \frac{1}{4}\left(2\langle x, y\rangle + 2\langle y, x\rangle\right) = \langle x, y\rangle$ (polar identity)

6.2(a) $S = \{ \underset{w_1}{\underset{\parallel}{(1,0,1)}}, \underset{w_2}{\underset{\parallel}{(0,1,1)}}, \underset{w_3}{\underset{\parallel}{(1,3,3)}} \}$.

$v_1 = w_1 = (1,0,1)$

$$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = (0,1,1) - \frac{1}{2}(1,0,1) = \left(-\frac{1}{2}, 1, \frac{1}{2}\right) = \frac{1}{2}(-1,2,1)$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 = \frac{\langle w_3, 2v_2\rangle}{\|2v_2\|^2}(2v_2)$$

$$= (1,3,3) - \frac{4}{2}(1,0,1) - \frac{8}{6}(-1,2,1)$$

$$= \left(\frac{1}{3}, \frac{1}{3}, -\frac{1}{3}\right) = \frac{1}{3}(1,1,-1)$$

$$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{2}}(1,0,1)$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{\sqrt{6}}(-1,2,1)$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{\sqrt{3}}(1,1,-1)$$

$\beta = \{u_1, u_2, u_3\}$ is an orthonormal basis for $\mathrm{span}(S) = \mathbb{R}^3$.

Fourier coefficients of $x = (1,1,2)$:

$$\langle x, u_1\rangle = \frac{1}{\sqrt{2}} \cdot 3, \quad \langle x, u_2\rangle = \frac{1}{\sqrt{6}} \cdot 3, \quad \langle x, u_3\rangle = \frac{1}{\sqrt{3}} \cdot 0 = 0$$

$$\Rightarrow x = \frac{3}{\sqrt{2}} u_1 + \frac{3}{\sqrt{6}} u_2 + 0 \cdot u_3$$

check: $\frac{3}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(1,0,1) + \frac{3}{\sqrt{6}} \cdot \frac{1}{\sqrt{6}}(-1,2,1) = \frac{3}{2}(1,0,1) + \underset{}{\overset{\frac{1}{2}}{\overset{\parallel}{\frac{3}{6}}}}(-1,2,1) = (1,1,2)$ ✓.

2(c) $V = P_2(\mathbb{R})$. $S = \{1, x, x^2\}$, $h(x) = 1 + x$.

$$\underset{w_1}{1},\ \underset{w_2}{x},\ \underset{w_3}{x^2}$$

$v_1 = w_1 = 1$, $\quad \|v_1\|^2 = \int_0^1 1 \cdot 1 dx = x\Big|_0^1 = 1$

$v_2 = w_2 - \frac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1$ $\quad \langle w_2, v_1\rangle = \int_0^1 x \cdot 1 dx = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}$

$$v_2 = x - \frac{\frac{1}{2}}{1} \cdot 1 = x - \frac{1}{2}$$

$$v_3 = w_3 - \frac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \frac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2$$

$\langle w_3, v_1\rangle = \int_0^1 x^2 \cdot 1 dx = \frac{1}{3}x^3\Big|_0^1 = \frac{1}{3}$.

$\langle w_3, v_2\rangle = \int_0^1 x^2(x - \frac{1}{2}) dx = \frac{1}{4}x^4 - \frac{1}{6}x^3\Big|_0^1 = \frac{1}{12}$.

$\|v_2\|^2 = \int_0^1 (x-\frac{1}{2})\cdot(x - \frac{1}{2}) = \int_0^1 (x^2 - x + \frac{1}{4}) dx = \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right)\Big|_0^1 = \frac{4-6+3}{12} = \frac{1}{12}$

$$v_3 = x^2 - \frac{\frac{1}{3}}{1}\cdot 1 - \frac{\frac{1}{12}}{\frac{1}{12}}\cdot(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

$$\|v_3\|^2 = \int_0^1 (x^2 - x + \frac{1}{6})^2 dx = \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 + \frac{x^2}{3} - \frac{x}{3}\right) dx$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} + \frac{1}{9} - \frac{1}{6} = \frac{1}{180}(36 + 60 + 5 - 90 + 20 - 30) = \frac{1}{180}$$

$\Rightarrow u_1 = \frac{v_1}{\|v_1\|} = 1$, $\quad u_2 = \frac{v_2}{\|v_2\|} = \frac{x - \frac{1}{2}}{\frac{1}{\sqrt{12}}} = 2\sqrt{3}\cdot(x - \frac{1}{2}) = 2\sqrt{3}x - \sqrt{3}$.

$u_3 = \frac{v_3}{\|v_3\|} = \frac{x^2 - x + \frac{1}{6}}{\frac{1}{6\sqrt{5}}} = 6\sqrt{5}x^2 - 6\sqrt{5}x + \sqrt{5}$.

$\beta = \{u_1, u_2, u_3\}$ is an orthonormal basis for $P_2(\mathbb{R})$.

Fourier coefficients for h relative to β:

$a_i = \langle h, u_i \rangle = \frac{\langle h, v_i \rangle}{\|v_i\|}$, $h = a_1 u_1 + a_2 u_2 + a_3 u_3$

$\langle h, v_1 \rangle = \int_0^1 (1+x)\cdot 1 \, dx = \left(x + \frac{1}{2}x^2\right)\Big|_0^1 = \frac{3}{2}$

$\langle h, v_2 \rangle = \int_0^1 (1+x)\cdot\left(-\frac{1}{2}+x\right)dx = \int_0^1 \left(x^2 + \frac{1}{2}x - \frac{1}{2}\right)dx = \frac{1}{3} + \frac{1}{4} - \frac{1}{2} = \frac{1}{12}$.

$\langle h, v_3 \rangle = \int_0^1 (1+x)\cdot\left(\frac{1}{6} - x + x^2\right)dx = \int_0^1 \left(\frac{1}{6} - \frac{5}{6}x + \underbrace{x^2\cdot(1-1)}_{=0} + x^3 \cdot 1\right) dx$

$= \left(\frac{1}{6}x - \frac{5}{12}x^2 + \frac{1}{4}x^4\right)\Big|_0^1 = \frac{1}{12}(2-5+3) = 0$.

$\Rightarrow a_1 = \frac{\frac{3}{2}}{1} = \frac{3}{2}$, $a_2 = \frac{\frac{1}{12}}{\frac{1}{\sqrt{12}}} = \frac{1}{\sqrt{12}} = \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{6}$, $a_3 = \frac{0}{\|v_3\|} = 0$.

$\Rightarrow$ $h = \frac{3}{2}\cdot u_1 + \frac{\sqrt{3}}{6}\cdot u_2 + 0 \cdot u_3$

Check: $\frac{3}{2}\cdot 1 + \frac{\sqrt{3}}{6}\cdot(2\sqrt{3}x - \sqrt{3}) = x + 1 = h(x)$ ✓.