4(d): $A=\begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix}$

- Find eigenvalues:

$$|A-\lambda I| = \begin{vmatrix} -\lambda & -3 & 1 & 2 \\ -2 & 1-\lambda & -1 & 2 \\ -2 & 1 & -1-\lambda & 2 \\ -2 & -3 & 1 & 4-\lambda \end{vmatrix} = \begin{vmatrix} 0 & (1-\lambda)\cdot(-\frac{\lambda}{2})-3 & \frac{\lambda}{2}+1 & -\lambda+2 \\ -2 & 1-\lambda & -1 & 2 \\ 0 & \lambda & -\lambda & 0 \\ 0 & \lambda-4 & 2 & 2-\lambda \end{vmatrix}$$

$$= -(-2)\cdot \begin{vmatrix} \frac{1}{2}\lambda^2-\frac{\lambda}{2}-3 & \frac{1}{2}(\lambda+2) & -(\lambda-2) \\ \lambda & -\lambda & 0 \\ \lambda-4 & 2 & 2-\lambda \end{vmatrix} = 2\cdot\lambda\cdot(\lambda-2) \begin{vmatrix} \frac{1}{2}(\lambda^2-\lambda-6) & \frac{1}{2}(\lambda+2) & -1 \\ 1 & -1 & 0 \\ \lambda-4 & 2 & -1 \end{vmatrix}$$

$$= 2\lambda\cdot(\lambda-2)\cdot \begin{vmatrix} \overset{\frac{1}{2}(\lambda+2)(\lambda-2)}{\frac{1}{2}(\lambda^2-4)} & \frac{1}{2}(\lambda+2) & -1 \\ 0 & -1 & 0 \\ \lambda-2 & 2 & -1 \end{vmatrix} = 2\lambda\cdot(\lambda-2)\cdot(-1)\cdot(\lambda-2)\cdot \begin{vmatrix} \frac{1}{2}(\lambda+2) & -1 \\ 1 & -1 \end{vmatrix}$$

$$= -2\lambda\cdot(\lambda-2)^2\cdot\big(\underbrace{-\tfrac{1}{2}(\lambda+2)+1}_{=-\lambda}\big) = \lambda^2\cdot(\lambda-2)^2 = 0.$$

$$\Rightarrow \lambda=0,\ m_0=2;\ \lambda=2,\ m_2=2$$

- Find Jordan blocks for each eigenvalue:

$$\lambda=0:\quad A-0I \longrightarrow \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 2 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & -1 & 1 & -2 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow N(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \qquad \dim E_0 = 1 < m_0 = 2.$$

$\Rightarrow$ dot diagram for $\lambda = 0$: $\begin{matrix} v_1 & \bullet \\ \uparrow & \\ v_2 & \bullet \end{matrix}$ $\Leftrightarrow$ 1 Jordan block: $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Find $v_2 \in N((A-0I)^2) - N(A-0I)$:

$$(A-0I)^2 = \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{pmatrix} = \begin{pmatrix} 0 & -8 & 4 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & 0 & 0 & 4 \\ -4 & -8 & 4 & 8 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 0 & 2 & -1 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -8 & 4 & 4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow N(A^2) = \text{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \\ 2 \end{pmatrix} \right\}$$

$\rightsquigarrow$ choose $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$

$$\Rightarrow v_1 = A\cdot v_2 = 1\cdot\begin{pmatrix}-3\\1\\1\\-3\end{pmatrix} + 2\cdot\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\-1\\-1\end{pmatrix} \in N(A-0\cdot I).$$

- $\lambda = 2$:

$$A-2I = \begin{pmatrix}-2&-3&1&2\\-2&-1&-1&2\\-2&1&-3&2\\-2&-3&1&2\end{pmatrix} \to \begin{pmatrix}2&3&-1&-2\\0&2&-2&0\\0&4&-4&0\\0&0&0&0\end{pmatrix} \to \begin{pmatrix}2&0&2&-2\\0&1&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

$$\to \begin{pmatrix}1&0&1&-1\\0&1&-1&0\\0&0&0&0\\0&0&0&0\end{pmatrix} \Rightarrow N(A-2I) = \mathrm{Span}\left\{ \underset{v_3}{\underset{\parallel}{\begin{pmatrix}-1\\1\\1\\0\end{pmatrix}}},\ \underset{v_4}{\underset{\parallel}{\begin{pmatrix}1\\0\\0\\1\end{pmatrix}}} \right\}$$

$\Rightarrow \dim E_2 = 2 = m_2 \Rightarrow$ dot diagram $\underset{v_3}{\bullet}\quad \underset{v_4}{\bullet}$

$\Leftrightarrow$ 2 Jordan blocks of size 1

- The Jordan canonical form of A is

$$J = \begin{pmatrix}\boxed{\begin{matrix}0&1\\0&0\end{matrix}} & & \\ & \boxed{2} & \\ & & \boxed{2}\end{pmatrix}. \qquad Q = (v_1\ v_2\ v_3\ v_4) = \begin{pmatrix}-1&0&-1&1\\-1&1&1&0\\-1&2&1&0\\-1&0&0&1\end{pmatrix}$$

satisfies $Q^{-1}\cdot A\cdot Q = J$.

5(c): $T: P_3(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$, $T(f(x)) = f''(x) + 2f(x)$.

- Choose standard basis $\alpha = \{1, x, x^2, x^3\}$.

$$[T] = \Big([T(1)]_\alpha \quad [T(x)]_\alpha \quad [T(x^2)]_\alpha \quad [T(x^3)]_\alpha \Big)$$

$$\begin{array}{cccc} \parallel & \parallel & \parallel & \parallel \\ [2]_\alpha & [2x]_\alpha & [2+2x^2]_\alpha & [6x+2x^3]_\alpha \end{array}$$

$$= \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 6 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} = A.$$

- $\det(A - \lambda I) = (2-\lambda)^4 = 0 \Rightarrow \lambda = 2$, $m_2 = 4$.

- $A - 2I = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$\Rightarrow N(A-2I) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ $\quad \dim E_2 = 2 < 4 = m_2$

- $(A-2I)^2 = \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = O_{4\times 4}$

$\Rightarrow N((A-2I)^2) = \mathbb{R}^4$, $\dim N((A-2I)^2) = 4 = 2 + 2$

# of dots on 2nd row

$\Rightarrow$ dot diagram $\quad \begin{array}{cc} v_1 \bullet & \bullet\, v_3 \\ v_2 \bullet & \bullet\, v_4 \end{array}$

choose $v_2, v_4 \in N((A-2I)^2) - N(A-2I)$:

$$v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \quad v_1 = (A-2I)v_2 = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = (A-2I)v_4 = \begin{pmatrix} 0 \\ 6 \\ 0 \\ 0 \end{pmatrix}$$

- Jordan canonical form of T: $J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} & \\ & \boxed{\begin{matrix} 2 & 1 \\ 0 & 2 \end{matrix}} \end{pmatrix}$

Jordan canonical basis :

$$\begin{Bmatrix} 2, & x^2, & 6x, & x^3 \\ \Updownarrow & \Updownarrow & \Updownarrow & \Updownarrow \\ v_1 & v_2 & v_3 & v_4 \end{Bmatrix}$$

verify: $Tv_1 = T(2) = 0 + 2\cdot 2 = 2v_1$

$Tv_2 = T(x^2) = 2 + 2x^2 = v_1 + 2v_2$

$Tv_3 = T(6x) = 0 + 2\cdot 6x = 2v_3$

$Tv_4 = T(x^3) = 6\cdot x + 2\cdot x^3 = 6v_3 + 2v_4$ ✓

6. · $\det(A-\lambda I) = \det(A^t - \lambda I)$

$\Rightarrow$ A and A^t have the same characteristic polynomial.

$\Rightarrow$ A and A^t have the same eigenvalues with same multiplicities

· For any eigenvalue λ, and any positive integer r

$$\operatorname{rank}\left((A-\lambda I)^r\right) = \operatorname{rank}\left(\left((A-\lambda I)^r\right)^t\right) = \operatorname{rank}\left((A^t - \lambda I)^r\right)$$

$$\Rightarrow \quad \operatorname{nullity}\left((A-\lambda I)^r\right) = \operatorname{nullity}\left((A^t-\lambda I)^r\right).$$

Assuming the characteristic polynomial splits

$\Rightarrow$ A and A^t have the same dot diagram

$\Rightarrow$ A and A^t have the same Jordan canonical form J

$\Rightarrow$ $A \sim J \sim A^t$: A and A^t are similar $\blacksquare$