- $T: V \to W$ linear transformation.

$\beta = \{v_1, v_2, \cdots, v_n\}$, $\quad \gamma = \{w_1, w_2, \cdots, w_m\}$

$$[T]_\beta^\gamma = \left([T(v_1)]_\gamma \ [T(v_2)]_\gamma \cdots [T(v_n)]_\gamma \right).$$

$$V \ni v = a_1 \cdot v_1 + a_2 \cdot v_2 + \cdots + a_n \cdot v_n \rightsquigarrow [v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{R}^n.$$

$$Tv = a_1 \cdot T(v_1) + a_2 \cdot T(v_2) + \cdots + a_n \cdot T(v_n)$$

$$\Rightarrow [Tv]_\gamma = a_1 \cdot [T(v_1)]_\gamma + a_2 \cdot [T(v_2)]_\gamma + \cdots + a_n \cdot [T(v_n)]_\gamma$$

$$= \left([T(v_1)]_\gamma \ [T(v_2)]_\gamma \cdots [T(v_n)]_\gamma \right) \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = [T]_\beta^\gamma \cdot [v]_\beta.$$

<u>Ex</u>: $T: P_2(\mathbb{R}) \longrightarrow P_3(\mathbb{R})$

$$f \longmapsto f(x) \cdot (x-1) + \int_0^x f(t)\,dt$$

$\beta = \{1, x, x^2\}$, $\quad \gamma = \{1, x, x^2, x^3\}$

$$T(1) = 1 \cdot (x-1) + \int_0^x 1 \cdot dt = x - 1 + x = 2x - 1$$

$$T(x) = x \cdot (x-1) + \int_0^x t\,dt = x^2 - x + \frac{x^2}{2} = \frac{3}{2}x^2 - x$$

$$T(x^2) = x^2 \cdot (x-1) + \int_0^x t^2\,dt = x^3 - x^2 + \frac{1}{3}x^3 = \frac{4}{3}x^3 - x^2$$

$$\Rightarrow [T]_\beta^\gamma = \left([T(1)]_\gamma \ [T(x)]_\gamma \ [T(x^2)]_\gamma \right) = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix}$$

Let $g = x^2 - x + 2$. Then

$$[Tg]_\gamma = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -1 & 0 \\ 0 & \frac{3}{2} & -1 \\ 0 & 0 & \frac{4}{3} \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \\ -\frac{5}{2} \\ \frac{4}{3} \end{pmatrix} \Rightarrow Tg = -2 + 5x - \frac{5}{2}x^2 + \frac{4}{3}x^3$$

$$\parallel$$

$$-2 + 5x - \frac{5}{2}x^2 + \frac{4}{3}x^3$$

$$\parallel$$

Verify: $Tg = (x^2 - x + 2) \cdot (x-1) + \int_0^x (t^2 - t + 2)\,dt$

$$= (x^3 - x^2 + 2x) - (x^2 - x + 2) + \frac{x^3}{3} - \frac{x^2}{2} + 2x$$

Let $T: V \to W$ be a linear transformation

<u>Def</u>: A function $U: W \to V$ is called an inverse of T if
$UT = Id_V$ and $TU = Id_W$.

T is called invertible if there is an inverse of T.

<u>Thm</u>: T is invertible if and only if T is $\underbrace{\text{both one-to-one and onto.}}_{\Updownarrow \text{ bijective.}}$

<u>Pf</u>: "only if" Use the basic facts:

$V \xrightarrow{T} W \xrightarrow{U} Z$ the composition UT is one-to-one $\Rightarrow T$ is one-to-one
onto $\Rightarrow U$ is onto.

$UT = Id_V \Rightarrow T$ is one-to-one. $TU = Id_W \Rightarrow T$ is onto

"if" one-to-one + onto $\Rightarrow$ bijective $\Rightarrow$ $\exists$ an inverse $U: W \to V$ s.t.
$U(\underbrace{T(v)}_{= w}) = v$. $\forall v \in V$. ∎

<u>Thm</u>: If T is invertible, then there is a unique inverse $T^{-1}: W \to V$.
Moreover T^{-1} is linear.

<u>Pf</u>: Uniqueness is easy to see: $T^{-1}(T(v)) = v$ defines T^{-1} uniquely.
<u>T^{-1} is linear</u>: $\forall w_1, w_2 \in W$, Let $v_1 = T^{-1}(w_1)$, $v_2 = T^{-1}(w_2)$.
Then $T^{-1}(w_1 + w_2) = T^{-1}(T(v_1) + T(v_2)) = T^{-1}(T(v_1 + v_2)) = v_1 + v_2 = T^{-1}(w_1) + T^{-1}(w_2)$
$T^{-1}(c \cdot w_1) = T^{-1}(c \cdot T(v_1)) = T^{-1}(T(cv_1)) = c \cdot v_1 = c \cdot T^{-1}(w_1)$

Assume V and W are finite dimensional.

Thm: $T: V \to W$ invertible $\Rightarrow \dim V = \dim W$.

Pf: T one-to-one $\Rightarrow N(T) = \{0\} \Rightarrow \text{nullity}(T) = 0$.

(Dimension Formula: $\text{nullity}(T) + \text{rank}(T) = \dim V$)

$\Rightarrow \underbrace{\text{rank}(T)}_{\dim R(T) \leq \dim W} = \dim(V) - \text{nullity}(T) = \dim V$.

T one-to-one $\Rightarrow \underline{\dim V \leq \dim W}$

T is onto $\Rightarrow R(T) = W \Rightarrow \text{rank}(T) = \dim W$

$\Rightarrow \underbrace{\text{rank}(T)}_{\dim W} = \dim(V) - \text{nullity}(T) \leq \dim V$

T onto $\Rightarrow \underline{\dim W \leq \dim V}$

So $\dim V = \dim W$. ∎

Thm: $T: V \to W$ linear. Assume $\dim V = \dim W$.

The following conditions are equivalent:

(1) T is invertible

(2) T is one-to-one

(3) T is onto.

Pf: (1) $\Leftrightarrow$ (2) + (3)

(2) T is one-to-one $\Rightarrow \text{nullity}(T) = 0 \Rightarrow \text{rank}(T) = \underbrace{\dim V}_{\dim V} - \text{nullity}(T)$

(2) $\Downarrow$ (1) $\xrightarrow{\dim V = \dim W} \underbrace{\text{rank}(T)}_{\dim R(W)} = \dim W \Rightarrow R(W) = W$ so. T is onto.

(3) T onto $\Rightarrow \text{rank}(T) = \dim W = \dim V \Rightarrow \text{nullity}(T) = 0 \Rightarrow N(T) = 0 \Leftrightarrow T$ is one-to-one. ∎

Ex: $T: \mathbb{R}^3 \to \mathbb{R}^3 \quad (a_1, a_2, a_3) \mapsto (2a_1 + 4a_3,\ a_2,\ a_2 - a_3)$

Determine whether T is invertible.

Sol: By the above thm, it is enough to check if $N(T) = \{0\}$.

$$(a_1, a_2, a_3) \in N(T) \Leftrightarrow \begin{cases} 2a_1 + 4a_3 = 0 \\ a_2 = 0 \\ a_2 - a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases} \Rightarrow N(T) = \{0\}.$$

So T is invertible.

- **Def:** V is called isomorphic to W, denoted by $V \cong W$, if there exists an invertible linear transformation $T: V \to W$.

Thm: Assume V and W are finite dimensional. Then

$$V \cong W \Leftrightarrow \dim V = \dim W.$$

In fact, if $\dim V = n$, then $V \cong \mathbb{R}^n \cong W$

Pf: - Choose a basis for V: $\beta = \{v_1, \cdots, v_n\}$ $\quad n = \dim V$

Consider the function

$$\begin{array}{ccc} T: & V & \longrightarrow & \mathbb{R}^n \\ & \cup\!\!| & & \cup\!\!| \\ a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = & v & \mapsto & [v]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \end{array}$$

Then T is an invertible linear transformation. So $V \cong \mathbb{R}^n$.

- If $T: V \to W$ is invertible and $U: W \to Z$ is invertible, then $UT: V \to Z$ is invertible. So

$$\left.\begin{array}{l} V \cong W \\ W \cong Z \end{array}\right\} \Rightarrow \underset{\substack{\cong \\ W}}{V} \cong Z.$$

In particular $\dim V = \dim W = n$

$$\Downarrow$$

$$V \cong W \cong \mathbb{R}^n$$

Ex: $P_n(\mathbb{R}) = \{\text{polynomials of degree} \leq n\} \cong \mathbb{R}^{n+1}$

$M_{n\times n}(\mathbb{R}) = \{n\times n \text{ matrices}\} \cong \mathbb{R}^{n^2}$

- $T: \underset{\beta}{\overset{\uparrow}{V}} \to \underset{\gamma}{\overset{\uparrow}{W}}$ invertible $\rightsquigarrow$ inverse $T^{-1}: W \to V$

$$\Rightarrow [T]_\beta^\gamma \cdot [T^{-1}]_\gamma^\beta = [T\cdot T^{-1}]_\gamma^\gamma = [Id_W]_\gamma^\gamma = I_{n\times n}$$

$n = \dim W = \dim V$

$$[T^{-1}]_\gamma^\beta \cdot [T]_\beta^\gamma = [T^{-1}\cdot T]_\beta^\beta = [Id_V]_\beta^\beta = I_{n\times n}.$$

$(n\times n)$ identity matrix

Def: A $n\times n$ matrix is invertible if $\exists$ $n\times n$ matrix B s.t. $AB = BA = I_{n\times n}$.