Quiz 5. $A=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{pmatrix}$.

1. Find the Jordan canonical basis β for A.

2. Use Gram-Schmidt to get an o.n.b. from β.

Sol: 1. $|A-\lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 = 0 \Rightarrow \lambda=1,\ m_1=3.$

$A-1\cdot I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_1 = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. $\dim E_1 = 2$

$\Rightarrow$ dot diagram $\quad v_1 \bullet \quad \bullet\, v_3$

$\qquad\qquad\qquad\qquad v_2 \bullet$

There are two ways to proceed:

Way 1: Find $v_2 \in N((A-I)^2) - N(A-I)$.

$(A-I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N((A-I)^2) = \mathbb{R}^3$.

Choose $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow v_1 = (A-I)v_2 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$. and choose $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

Way 2: Find $v_1 \in R(A-I) \cap N(A-I)$.

$R(A-I) = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \right\}$. Choose $v_1 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix} \in R(A-I) \cap N(A-I)$.

$v_1 = (A-I)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \Rightarrow$ Choose $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$. Choose $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$.

2. $v_1 = \begin{pmatrix} 0 \\ 0 \\ -2 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ are orthogonal to each other.

$\Rightarrow$ Gram-Schmidt orthogonalization process produce the o.n.b.:

$$\left\{ u_1 = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix},\ u_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$