

$$A = \begin{pmatrix} 1 & 2 & 1 & -2 & 1 \\ -1 & -2 & -1 & 2 & 0 \\ 1 & 2 & -1 & 4 & 1 \end{pmatrix}$$

(1) Find rank(A) and nullity(A)

(2) Find a basis for the column space of A.

(3) Find a basis for the null space $N(A)$.

Sol: (1) Gauss elimination:

$$\begin{pmatrix} 1 & 2 & 1 & -2 & 1 \\ -1 & -2 & -1 & 2 & 0 \\ 1 & 2 & -1 & 4 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 6 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 1 & -2 & 1 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 2 & 1 & -2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$\uparrow \quad \uparrow \quad \uparrow$
 1st 3rd 5th

$$\text{rank}(A) = \# \text{ leading variables} = 3$$

$$\text{nullity}(A) = 5 - \text{rank}(A) = 5 - 3 = 2$$

(2) basis for column space:

$$\left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$\uparrow \quad \uparrow \quad \uparrow$
 1st 3rd 5th

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

$$(3) \quad N(A) = \{ \overset{11}{x} \in \mathbb{R}^5 : Ax = 0 \}$$

$$Ax = 0 \Leftrightarrow \text{rref}(A)x = 0 \Leftrightarrow \begin{cases} x_1 + 2x_2 + x_4 = 0 \\ x_3 - 3x_4 = 0 \\ x_5 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x_1 = -2x_2 - x_4 \\ x_3 = 3x_4 \\ x_5 = 0 \end{cases}$$

(2)

$$\Leftrightarrow x = \begin{pmatrix} -2x_2 - x_4 \\ x_2 \\ 3x_4 \\ x_4 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}$$

$$\text{basis for } N(A): \left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix} \right\}$$

(2)