Quiz 2:

Q: Is each of the following subset a basis for $P_2(\mathbb{R})$? Why?

(1) $S_1 = \{1+x+x^2, 2+6x^2\}$

(2) $S_2 = \{1+x+x^2, 2+6x^2, 1-x+5x^2\}$

(3) $S_3 = \{1+x+x^2, 2+6x^2, 1-x^2\}$.

Solution: $\beta = \{1, x, x^2\}$ is a basis for $P_2(\mathbb{R})$

$\Rightarrow \dim P_2(\mathbb{R}) = \#\beta = 3$

$\Rightarrow$ any basis of $P_2(\mathbb{R})$ has exactly 3 vectors (polynomials).

(1) $\#S_1 = 2 < 3 \Rightarrow S_1$ is not a basis.

(2) $\#S_2 = 3$. S_2 is a basis $\Leftrightarrow$ S_2 is linearly independent.

Find linear relations of S_2:

$$a_1(1+x+x^2) + a_2(2+6x^2) + a_3(1-x+5x^2) = 0$$

$$\parallel$$

$$(a_1+2a_2+a_3) + (a_1-a_3)x + (a_1+6a_2+5a_3)$$

$$\Leftrightarrow \begin{cases} a_1+2a_2+a_3=0 \Rightarrow 2a_1+2a_2=0 \\ \qquad\qquad\qquad\qquad\quad \Uparrow \\ a_1-a_3=0 \Rightarrow a_3=a_1 \\ \qquad\qquad\qquad\qquad\quad \Downarrow \\ a_1+6a_2+5a_3=0 \Rightarrow 6a_1+6a_2=0 \end{cases} \Leftrightarrow \begin{cases} a_2=-a_1 \\ a_3=a_1 \end{cases}$$

$\Rightarrow$ nontrivial linear relations when $a_1 \neq 0$. For example, $a_1=1$:

$$(1+x+x^2) - (2+6x^2) + (1-x+6x^2) = 0$$

$\Rightarrow S_2$ is linear dependent $\Rightarrow S_2$ is not a basis.

<u>OR</u>: Use Gauss elimination:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 1 & 6 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -2 & -2 \\ 0 & 4 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{cases} a_1 = a_3 \\ a_2 = -a_3 \end{cases}$$

(3) By the same reason as (2), we look for linear relations:

$$a_1 \cdot (1 + x + x^2) + a_2 \cdot (2 + 6x^2) + a_3 (1 - x^2) = 0$$

$$\parallel$$

$$(a_1 + 2a_2 + a_3) + a_1 \cdot x + (a_1 + a_2 - a_3) = 0$$

$$\Leftrightarrow \begin{cases} a_1 + 2a_2 + a_3 = 0 \\ a_1 = 0 \\ a_1 + a_2 - a_3 = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 = 0 \\ 2a_2 + a_3 = 0 \\ a_2 - a_3 = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 . \end{cases}$$

$$\left.\begin{array}{l} \Rightarrow S_3 \text{ is linearly independent} \\ \# S_3 = 3 = \dim P_2(\mathbb{R}) \end{array}\right\} \Longrightarrow S_3 \text{ is a basis for } P_2(\mathbb{R}).$$

<u>Another method for part (3)</u>:

Check whether Span S_3 = Span$\{1, x, x^2\}$ = $P_2(\mathbb{R})$

$$1 = (2+6x^2+6(1-x^2))\frac{1}{8}$$
$$= \frac{1}{8}(2+6x^2)+\frac{6}{8}(1-x^2) \quad \Rightarrow 1 \in \text{Span } S_3$$

$$x^2 = (2+6x^2-2(1-x^2))\frac{1}{8} \quad \Rightarrow x^2 \in \text{Span } S_3$$
$$= \frac{1}{8}(2+6x^2)-\frac{2}{8}(1-x^2).$$

$$x = (1+x+x^2)-1-x^2$$
$$= (1+x+x^2)-\underbrace{\left(\frac{1}{8}(2+6x^2)+\frac{6}{8}(1-x^2)\right)-\left(\frac{1}{8}(2+6x^2)-\frac{2}{8}(1-x^2)\right)}_{\in \text{ Span } S_3}$$

$$\Rightarrow \left.\begin{array}{l} \text{Span}(S_3) = P_2(\mathbb{R}) \\ \# S_3 = 3 \end{array}\right\} \Rightarrow S_3 \text{ is a basis for } P_2(\mathbb{R}).$$