!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : ID :

1(20pts) Let $S = \{v_1, v_2, v_3, v_4\}$ be a subset of $\mathbf{R}^3$ with

$$v_1 = (1,0,1), v_2 = (0,1,0), v_3 = (1,-1,1), v_4 = (1,0,-1).$$

(1) Is S a basis for $\mathbf{R}^3$? Why?
(2) Find a subset β of S such that β is a basis for $\mathbf{R}^3$.

(1) $\#S = 4 > 3 = \dim \mathbb{R}^3 \Rightarrow S$ is not a basis

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Indeed: $v_3 = v_1 - v_2 \Rightarrow S$ is not linearly independent.

(2) Construct a basis by adjoining linearly independent vectors:

$v_1 \neq 0 \Rightarrow \{v_1\}$ linearly independent $\xRightarrow{v_2 \notin \operatorname{Span}\{v_1\}}$ $\{v_1, v_2\}$ is linearly independent

$\xRightarrow{v_3 \in \operatorname{Span}\{v_1, v_2\}}$ do not include v_3 $\xRightarrow{v_4 \notin \operatorname{Span}\{v_1, v_2\}}$ $\{v_1, v_2, v_4\}$ linearly indep.

$\Rightarrow \beta = \{v_1, v_2, v_4\}$ is a basis.

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show: $\{v_1, v_2, v_4\}$ is linearly independent: assume $av_1 + bv_2 + cv_4 = 0$

Then $a\cdot(1,0,1) + b(0,1,0) + c(1,0,-1) = 0 = (a+c, b, a-c)$

$$\Updownarrow$$

$$\begin{cases} a + c = 0 \\ b = 0 \\ a - c = 0 \end{cases} \Rightarrow a = b = c = 0$$ ✓

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2(30pts) Consider the linear transformation:

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R}), \quad T(f) = xf'(x) + f(x).$$

Let $\beta = \{1, x, x^2\}$, $\gamma = \{1, 1+x, x+x^2\}$.

(1) Find the matrix representation $[T]_\gamma^\beta$

(2) Find the matrix representation $[T]_\beta^\gamma$.

(3) Let $f(x) = 1 + x^2$. Calculate $[T(f(x))]_\gamma$.

(1) $[T]_\gamma^\beta = \left([T(1)]_\beta \quad [T(1+x)]_\beta \quad [T(x+x^2)]_\beta \right)$

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$T(1) = 1\cdot 0 + 1 = 1$, $T(1+x) = x\cdot 1 + 1 + x = 1 + 2x$ 3

$T(x+x^2) = x\cdot(1+2x) + x + x^2 = 2x + 3x^2$ 3

$\Rightarrow [T]_\gamma^\beta = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$

(2) We can write vectors in β as linear combinations of γ: 3

$1 = 1$, $x = (-1)1 + 1\cdot(1+x)$, $x^2 = -x + (x+x^2) = 1\cdot 1 - 1\cdot(1+x) + 1\cdot(x+x^2)$

$\Rightarrow [Id]_\beta^\gamma = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$

$$[T]_\beta^\beta = \left(\underset{\substack{\| \\ [1]_\beta \\ \| \\ \begin{pmatrix}1\\0\\0\end{pmatrix}}}{[T(1)]_\beta} \quad \underset{\substack{\| \\ [2x]_\beta \\ \| \\ \begin{pmatrix}0\\2\\0\end{pmatrix}}}{[T(x)]_\beta} \quad \underset{\substack{\| \\ [x\cdot 2x + x^2]_\beta \\ \| \\ \begin{pmatrix}0\\0\\3\end{pmatrix}}}{[T(x^2)]_\beta} \right)$$

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$$\begin{cases} T(1) = 1 = 1\cdot 1 \\ T(x) = 2x = -2\cdot 1 + 2\cdot(1+x) \\ T(x^2) = 3x^2 = 3\cdot 1 + (-3)\cdot(1+x) + 3\cdot(x+x^2) \end{cases}$$

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$\Updownarrow$ 3

$$[T]_\beta^\gamma = [Id]_\beta^\gamma \cdot [T]_\beta^\beta = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{pmatrix}$$

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$\| \; [T(1)]_\gamma \quad \| \; [T(x)]_\gamma \quad \| \; [T(x^2)]_\gamma$

(3) $f(x) = 1 + x^2$

$$[T(f(x))]_\gamma = [T]_\beta^\gamma \cdot [f(x)]_\beta$$

$$= \begin{pmatrix} 1 & -2 & 3 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \quad 3$$

$$= \begin{pmatrix} 4 \\ -3 \\ 3 \end{pmatrix}. \quad 3$$

Verify: $T(f(x)) = x \cdot 2x + (1 + x^2) = 1 + 3x^2$.

$$// \quad ✓$$

$$4 \cdot 1 + (-3) \cdot (1 + x) + 3 \cdot (x + x^2) = 1 + 3x^2$$

3(20 pts) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined as:

$$T(x, y, z) = (x + y + z, x + 2y + 3z, x + 5y + 9z).$$

Determine whether T is one-to-one or onto.

We calculate $N(T)$:

$$\begin{cases} x+y+z=0 \\ x+2y+3z=0 \\ x+5y+9z=0 \end{cases} \Longleftrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 9 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

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Gauss elimination:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 5 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 8 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \Leftrightarrow \begin{cases} x = z \\ y = -2z \end{cases}$$

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$$\Rightarrow N(T) = \left\{ \begin{pmatrix} z \\ -2z \\ z \end{pmatrix}, z \in \mathbb{R} \right\} = \mathbb{R} \cdot \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \neq \{\underline{0}\} \qquad \text{nullity}(T) = 1$$

For example, $(1,-2,1) \overset{T}{\longmapsto} (1-2+1,\ 1+2\cdot(-2)+3\cdot 1,\ 1+5\cdot(-2)+9\cdot 1)$

$$\overset{\parallel}{(0,0,0)}$$

$\Rightarrow$ ~~N~~ T is not one-to-one

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$\dim R(T) = \text{Rank}(T) = \dim V - \text{nullity}(T) = 3 - 1 = 2 < \dim W$

$\Longrightarrow R(T) \subsetneq W$. T is not onto.

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4(30pts) Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces. Let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V.

(1) Assume that there a linear transformation $S : W \to V$ such that $T \circ S = \mathrm{Id}_W$. Must T be one-to-one or onto? Prove your claim.

(2) Assume that $\{T(v_1), \ldots, T(v_n)\}$ is a linearly independent subset of W. Must T be one-to-one or onto? Prove your claim.

(1) $T \circ S = Id_W \Rightarrow T$ must be onto. 5

Proof: For any $w \in W$, $T \circ S(w) = T(S(w)) = w$ 5

Set $v = S(w) \in V$. Then $T(v) = w$. So $w \in R(T)$ 5

So $R(T) = W \Rightarrow T$ is onto.

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(2) T must be one-to-one.

Proof: Assume $v', v'' \in V$ satisfy $T(v') = T(v'')$. 5

let $v' = a_1 \cdot v_1 + a_2 v_2 + \cdots + a_n \cdot v_n$ (β is a basis)

$v'' = b_1 \cdot v_1 + b_2 v_2 + \cdots + b_n \cdot v_n$

Then $T(v') = a_1 \cdot T(v_1) + a_2 \cdot T(v_2) + \cdots + a_n \cdot T(v_n)$

$\parallel$

$T(v'') = b_1 \cdot T(v_1) + b_2 \cdot T(v_2) + \cdots + b_n \cdot T(v_n)$

$\Rightarrow (a_1 - b_1) T(v_1) + (a_2 - b_2) T(v_2) + \cdots + (a_n - b_n) T(v_n) = 0$

Because $\{T(v_1), \ldots, T(v_n)\}$ linearly independent, $a_1 - b_1 = 0, a_2 - b_2 = 0, \ldots, a_n - b_n = 0$

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$\Rightarrow v' = v''$