!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME :

ID :

1(15pts) For each of the following subsets of $\mathbf{R}^3$, determine whether it is a vector subspace of $\mathbf{R}^3$. Explain your reason.

(1) $\{(a, b, c); a^2 + b^2 = c^2\}. = W_1$

(2) $\{(a, b, c); a + b = c\}. = W_2$

(3) $\{(a, b, c); \sin(a + b + c) = 0\}. = W_3$

(1) It is not a subspace because it is not closed under addition: $(0,2,\cancel{4}) \in W_1$, $(1,0,1) \in W_1$ but $(1,2,\cancel{5}) \notin W_1$

$1^2+2^2=5 \neq 5^2$

(2) It is a subspace because it is closed under addition and scalar multiplication:

$(a_1,b_1,c_1), (a_2,b_2,c_2) \in W_2 \Rightarrow a_1+b_1=c_1,\ a_2+b_2=c_2$

$\Downarrow$

$(a_1,b_1,c_1)+(a_2,b_2,c_2) \in W_2 \Leftarrow (a_1+a_2)+(b_1+b_2)=c_1+c_2$

$(a,b,c) \in W_2 \Rightarrow a+b=c \Rightarrow ka+kb=kc \Rightarrow k\cdot(a,b,c) \in W_2.$

(3) It is not a subspace because it is not closed under scalar multiplication: $(\pi,0,0) \in W_3$

$\frac{1}{2}(\pi,0,0) \notin W_3$; $\sin\left(\frac{\pi}{2}\right)=1 \neq 0$

2(20pts) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be a linear transformation that satisfies

$$T(1+x) = 1 - x, \quad T(1-x) = 1 + x, \quad T(2+x^2) = 2x - x^2.$$

Find the matrix representation of T with respect to the standard basis $\beta = \{1, x, x^2\}$.

Write vectors in β as linear combinations of $\{1+x, 1-x, 2+x^2\}$

$$1 = \frac{1}{2}(1+x) + \frac{1}{2}(1-x) \qquad 2$$

$$x = \frac{1}{2}(1+x) - \frac{1}{2}(1-x) \qquad 2$$

$$x^2 = -2 + (2+x^2) = -(1+x) - (1-x) + (2+x^2) \qquad 2$$

$$\Rightarrow T(1) = \frac{1}{2}T(1+x) + \frac{1}{2}T(1-x) = \frac{1}{2}(1-x) + \frac{1}{2}(1+x) = 1 \qquad 4$$

$$T(x) = \frac{1}{2}T(1+x) - \frac{1}{2}T(1-x) = \frac{1}{2}(1-x) - \frac{1}{2}(1+x) = -x \qquad 4$$

$$T(x^2) = -T(1+x) - T(1-x) + T(2+x^2) \qquad 4$$

$$= -(1-x) - (1+x) + 2x - x^2 = -2 + 2x - x^2$$

$$\Rightarrow [T]_\beta = \left([T(1)]_\beta \ [T(x)]_\beta \ [T(x^2)]_\beta\right) = \begin{pmatrix} 1 & 0 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{pmatrix}. \qquad 2$$

3(25pts) Consider the following subset S of $P_3(\mathbf{R})$.

$$S = \{1+x, 1-x^2, x+x^2+x^3, x^3\}$$

(1) Calculate the dimension of the subspace Span(S).
(2) Is the polynomial $1+4x+3x^2+x^3$ contained in Span(S)? If it is, write it as a linear combination of S.

(1)+(2) Under the standard basis $\beta = \{1, x, x^2\}$:

Consider the augmented matrix: $\begin{pmatrix} P_3(\mathbb{R}) \cong \mathbb{R}^4 \\ v \mapsto [v]_\beta \end{pmatrix}$

$$\left([1+x]_\beta \; [1-x^2]_\beta \; [x+x^2+x^3]_\beta \; [x^3]_\beta \;\middle|\; [1+4x+3x^2+x^3]_\beta \right)$$ 5

$$= \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 4 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right)$$ 5

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \leftarrow \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \text{consistent}$$ 5

$\Downarrow$

$1+4x+3x^2+x^3$ is contained in Span(S)

$$\downarrow$$

5 $$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow$$ a particular solution to inhomogeneous equation is $\begin{pmatrix} 3 \\ -2 \\ 1 \\ 0 \end{pmatrix}$

$\Rightarrow$ dim Span(S) = # leading variables = 3 5

$$1+4x+3x^2+x^3 = 3\cdot(1+x) - 2(1-x^2) + 1\cdot(x+x^2+x^3)$$

4(25pts) Consider the following linear transformation:

$$T : \mathbf{R}^3 \to \mathbf{R}^3, \quad T(a,b,c) = (c,b,a).$$

Determine whether T is diagonalizable. If it is, find a basis β such that $[T]_\beta$ is diagonal.

Let $\gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$ be the standard basis of $\mathbb{R}^3$.

$$[T]_\gamma = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = A \qquad 5$$

· characteristic polynomial: $f(t) = \det(A - tI) = \begin{vmatrix} -t & 0 & 1 \\ 0 & 1-t & 0 \\ 1 & 0 & -t \end{vmatrix}$

$$= -t \cdot \begin{vmatrix} 1-t & 0 \\ 0 & -t \end{vmatrix} + (-1)^{1+3} \begin{vmatrix} 0 & 1-t \\ 1 & 0 \end{vmatrix} = -t \cdot (1-t)(-t) + (-(1-t))$$

$$= t^2 \cdot (1-t) - (1-t) = (t^2-1)(1-t) = -(1-t)^2 \cdot (1+t) \qquad 5$$

$\Rightarrow$ eigenvalues $\lambda = -1$, $mult(-1) = 1$; $\lambda = 1$, $mult(1) = 2$. 5

· $\lambda = -1$: $A + I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_{-1} = \mathrm{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ 5

· $\lambda = 1$: $A - I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow E_1 = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$\dim N(A-I) = \dim E_1 = 2 = mult(1)$, $\dim N(A+I) = 1 = mult(-1)$

$\Rightarrow A$ is diagonalizable $\Rightarrow T = L_A$ is diagonalizable with respect to the basis $\beta = \{(-1,0,1), (0,1,0), (1,0,1)\}$. $[T]_\beta = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

5

5(15pts) Let r_1, r_2, r_3 be row vectors in $\mathbf{R}^3$. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 3r_1 + r_2 \\ r_3 \\ r_1 + 2r_2 \end{pmatrix} = k \cdot \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

(Hint: use the properties of determinant under row operations and the fact that it is linear for each row **when** other rows are fixed)

$$\det\begin{pmatrix} 3r_1+r_2 \\ r_3 \\ r_1+2r_2 \end{pmatrix} = -\det\begin{pmatrix} 3r_1+r_2 \\ r_1+2r_2 \\ r_3 \end{pmatrix} \quad ③$$

$$= -\det\begin{pmatrix} 3r_1+r_2 \\ r_1 \\ r_3 \end{pmatrix} - \det\begin{pmatrix} 3r_1+r_2 \\ 2r_2 \\ r_3 \end{pmatrix} \quad ③$$

$$= -\det\begin{pmatrix} 3r_1 \\ r_1 \\ r_3 \end{pmatrix} - \det\begin{pmatrix} r_2 \\ r_1 \\ r_3 \end{pmatrix} - \det\begin{pmatrix} 3r_1 \\ 2r_2 \\ r_3 \end{pmatrix} - \det\begin{pmatrix} r_2 \\ 2r_2 \\ r_3 \end{pmatrix} \quad ③$$

$$= 0 + \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - 6 \cdot \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} - 0 \quad ③$$

$$= -5 \cdot \det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} \quad \Rightarrow k = -5. \quad ③$$

6(25pts) Consider the following square matrix.

$$A = \begin{pmatrix} 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(1) Calculate the characteristic polynomial $f(t) = \det(A - tI)$ of A.
(2) Calculate the result of $A^4 - 2A^2$ (hint: use Cayley-Hamilton theorem).

(1) $f(t) = \begin{vmatrix} -t & 0 & 0 & 5 \\ 1 & -t & 0 & 0 \\ 0 & 1 & -t & 2 \\ 0 & 0 & 1 & -t \end{vmatrix} = (-t) \cdot \begin{vmatrix} -t & 0 & 0 \\ 1 & -t & 2 \\ 0 & 1 & -t \end{vmatrix} + (-1)^{1+4} \cdot 5 \cdot \begin{vmatrix} 1 & -t & 0 \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{vmatrix}$

$= (-t) \cdot (-t) \cdot \begin{vmatrix} -t & 2 \\ 1 & -t \end{vmatrix} - 5 \cdot 1 \cdot 1 \cdot 1 = t^2 \cdot (t^2 - 2) - 5 = t^4 - 2t^2 - 5.$

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(2) By Cayley-Hamilton theorem. $f(A) = 0$:

$A^4 - 2 \cdot A^2 - 5 \cdot I_4 = 0$

$\Rightarrow A^4 - 2A^2 = 5 I_4 = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix}$

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7(25pts) Let A be a square matrix with characteristic polynomial equal to $(t-2)^9$. Assume that the dot diagram of A is the following:

$$\begin{array}{cccc} \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & & \end{array}$$

(1) Write down the Jordan canonical form of A.
(2) Calculate $\dim N((A-2I)^2)$ and $\dim R((A-2I)^2)$.

(1) $J = \begin{pmatrix} \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 2 & 1 & \\ & 2 & 1 \\ & & 2 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}} & \\ & & & \boxed{2} \end{pmatrix}$

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(2) $\dim N((A-2I)^2) = \#$ dots in 1st row $+ \#$ dots in 2nd row,

$= 7$

5

$\dim R((A-2I)^2) = 9 - \dim N((A-2I)^2) = 9 - 7 = 2.$

$\uparrow$

A is a (9×9)-matrix

5

8(30pts) Consider the linear transformation:

$$T: P_2(\mathbf{R}) \to P_2(\mathbf{R}), \quad T(f(x)) = f''(x) + f'(x) + f(x).$$

(1) Find all eigenvalues of T and corresponding dot diagrams.
(2) Find a basis γ of $P_2(\mathbf{R})$ such that $[T]_\gamma$ is a Jordan canonical form.

(1) $[T]_\beta = \left(\underset{[1]_\beta}{\underset{\parallel}{[T(1)]_\beta}} \quad \underset{[1+x]_\beta}{\underset{\parallel}{[T(x)]_\beta}} \quad \underset{[2+2x+x^2]_\beta}{\underset{\parallel}{[T(x^2)]_\beta}} \right) = \overset{A}{\overset{\parallel}{\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}}}$ 5

$f(t) = \det(A - tI) = \begin{vmatrix} 1-t & 1 & 2 \\ 0 & 1-t & 2 \\ 0 & 0 & 1-t \end{vmatrix} = (1-t)^3 \Rightarrow$ eigenvalue $\lambda = 1$ with multiplicity $\text{mult}(1) = 3$.

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calculate $\dim N(A - tI)$:

$$(A - tI) = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A-I) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$\Rightarrow \dim N(A-I) = 1 \Rightarrow$ dot diagram:

- $(A-I)^2 v_1 = v_3$ 3
- $(A-I) v_1 = v_2$ 3
- v_1

(2) Calculate $N((A-tI)^2)$:

$$(A-I)^2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A-2I) = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

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choose $v_1 \in \underbrace{N((A-I)^3)}_{\overset{\parallel}{\mathbb{R}^3}} - N((A-I)^2)$

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \Rightarrow v_2 = (A-I)v_1 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}.$$

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$$\Rightarrow v_3 = (A-I)v_2 = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$ 5

$$\Rightarrow Q = \begin{pmatrix} v_3 & v_2 & v_1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ satisfies } Q^{-1}\cdot A\cdot Q = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$\Rightarrow$ basis $\gamma = \{2,\ 2+2x,\ x^2\}$. 5

<u>Check</u>: $T(2) = 2 = 1\cdot 2$

$T(2+2x) = 2 + (2+2x)$

$T(x^2) = 2 + 2x + x^2 = 1\cdot(2+2x) + 1\cdot x^2$ ✓

Alternatively, choose $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and solve $(A-I)v_2 = v_3$, $(A-I)v_1 = v_2$

$$((A-I)|v_3) = \left(\begin{array}{ccc|c} 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$(A-I|v_2) = \left(\begin{array}{ccc|c} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow v_1 = \begin{pmatrix} 0 \\ -1 \\ \frac{1}{2} \end{pmatrix}.$$

$\Rightarrow$ basis $\gamma = \{1,\ x,\ -x+\frac{1}{2}x^2\}$.

$T(1) = 1\cdot 1$, $T(x) = 1 + x = 1\cdot 1 + 1\cdot x$

$T(-x+\frac{1}{2}x^2) = 1 + (-1+x) + (-x+\frac{1}{2}x^2) = 1\cdot x + 1(-x+\frac{1}{2}x^2)$ ✓

9(20pts) Let $S : V \to W$ and $T : W \to U$ be linear transformations. Prove or disprove (i.e. find a counter-example to) each of the following statements for the composition $T \circ S : V \to U$.

(1) If T is onto, then $\text{rank}(T \circ S) = \text{rank}(S)$.
(2) If T is one-to-one, then $\text{rank}(T \circ S) = \text{rank}(S)$.

(1) Not true. $S: \mathbb{R}^2 \to \mathbb{R}^2 \quad S(x,y) = (x,y)$

$T: \mathbb{R}^2 \to \mathbb{R} \quad T(x,y) = x$

$\Rightarrow T \circ S(x,y) = T(x,y) = x$

$\Rightarrow \text{rank}(T \circ S) = 1 \neq \text{rank}(S) = 2$

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(2) True $\quad \text{rank}(T \circ S) = \dim(\underbrace{T \circ S(V)}_{R(T \circ S)}) = \dim T(S(V))$

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$\text{rank}(S) = \dim S(V)$

choose a basis for $S(V)$: $\beta = \{v_1, \cdots, v_n\}$ $<$ β is linearly independent; $\text{Span}(\beta) = S(V)$

It is enough to show that $\gamma = \{T(v_1), \cdots, T(v_n)\}$ is a basis for $T(S(V))$

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Prove: γ is linearly independent:

$0 = a_1 T(v_1) + \cdots + a_n T(v_n) = T(a_1 v_1 + \cdots + a_n v_n) \xrightarrow{T \text{ is one-to-one}} a_1 v_1 + \cdots + a_n v_n = 0$

β is linearly independent $\Longrightarrow a_1 = \cdots = a_n = 0$

. $\text{span}(\gamma)$: any element in $T(S(V))$ is of the form $T(w)$ where $w = a_1 v_1 + \cdots + a_n v_n$ because $\text{span}(\beta) = S(V)$.

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$$T(S(V)) \subseteq \mathrm{Span}(\gamma).$$

$\Uparrow$ w arbitrary in $S(V)$

$$\Rightarrow T(w) = T(a_1 v_1 + \cdots + a_n v_n)$$
$$= a_1 T(v_1) + \cdots + a_n T(v_n) \in \mathrm{Span}(\gamma)$$

On the other hand, $\mathrm{Span}(\gamma) \subseteq T(S(V))$ because any $u \in \mathrm{Span}(\gamma)$ is of the form

$$u = a_1 T(v_1) + \cdots + a_n T(v_n) = T(\underbrace{a_1 v_1 + \cdots + a_n v_n}_{\substack{\Uparrow \\ S(V)}}) \in T(S(V))$$

So we get $\mathrm{Span}(\gamma) = T(S(V))$.

So γ is indeed a basis for $T(S(V))$. $\blacksquare$