MATH 350 Spring 2023 FINAL EXAM

NAME: **ID:**

THERE ARE NINE (9) PROBLEMS. THEY HAVE THE INDICATED VALUE.

SHOW YOUR WORK

NO CALCULATORS NO CELLS ETC.

ON YOUR DESK: ONLY test, pen, pencil, eraser.

1		15pts
2		20pts
3		25pts
4		25pts
5		15pts
6		25pts
7		25pts
8		30pts
9		20pts
Total		200pts

!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(15pts) For each of the following subsets of $\mathbf{R}^3$, determine whether it is a vector subspace of $\mathbf{R}^3$. Explain your reason.

(1) $\{(a, b, c); a^2 + b^2 = c^2\}$.
(2) $\{(a, b, c); a + b = c\}$.
(3) $\{(a, b, c); \sin(a + b + c) = 0\}$.

2(20pts) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be a linear transformation that satisfies

$$T(1+x) = 1-x, \quad T(1-x) = 1+x, \quad T(2+x^2) = 2x - x^2.$$

Find the matrix representation of T with respect to the standard basis $\beta = \{1, x, x^2\}$.

3(25pts) Consider the following subset S of $P_3(\mathbf{R})$.

$$S = \{1 + x, 1 - x^2, x + x^2 + x^3, x^3\}$$

(1) Calculate the dimension of the subspace $\mathrm{Span}(S)$.

(2) Is the polynomial $1 + 4x + 3x^2 + x^3$ contained in $\mathrm{Span}(S)$? If it is, write it as a linear combination of S.

4(25pts) Consider the following linear transformation:

$$T : \mathbf{R}^3 \to \mathbf{R}^3, \quad T(a, b, c) = (c, b, a).$$

Determine whether T is diagonalizable. If it is, find a basis β such that $[T]_\beta$ is diagonal.

5(15pts) Let r_1, r_2, r_3 be row vectors in $\mathbf{R}^3$. Find the value of k that satisfies the following equation:

$$\det \begin{pmatrix} 3r_1 + r_2 \\ r_3 \\ r_1 + 2r_2 \end{pmatrix} = k \cdot \det \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}.$$

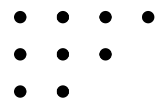
(Hint: use the properties of determinant under row operations and the fact that it is linear for each row **when** other rows are fixed)

6(25pts) Consider the following square matrix.

$$A = \begin{pmatrix} 0 & 0 & 0 & 5 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

(1) Calculate the characteristic polynomial $f(t) = \det(A - tI)$ of A.
(2) Calculate the result of $A^4 - 2A^2$ (hint: use Cayley-Hamilton theorem).

7(25pts) Let A be a square matrix with characteristic polynomial equal to $(t-2)^9$. Assume that the dot diagram of A is the following:



(1) Write down the Jordan canonical form of A.
(2) Calculate $\dim N((A-2I)^2)$ and $\dim R((A-2I)^2)$.

8(30pts) Consider the linear transformation:

$$T : P_2(\mathbf{R}) \to P_2(\mathbf{R}), \quad T(f(x)) = f''(x) + f'(x) + f(x).$$

(1) Find all eigenvalues of T and corresponding dot diagrams.
(2) Find a basis γ of $P_2(\mathbf{R})$ such that $[T]_\gamma$ is a Jordan canonical form.

9(20pts) Let $S : V \to W$ and $T : W \to U$ be linear transformations. Prove or disprove (i.e. find a counter-example to) each of the following statements for the composition $T \circ S : V \to U$.

(1) If T is onto, then $\text{rank}(T \circ S) = \text{rank}(S)$.

(2) If T is one-to-one, then $\text{rank}(T \circ S) = \text{rank}(S)$.

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