!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : **ID :**

1(25pts) A is a 3×6 matrix whose reduced row ethelon form is equal to:

$$\begin{pmatrix} 1 & 2 & 0 & -1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

(1) Find a basis for the null space $N(A)$.

(2) Assume that $A = (v_1\ v_2\ v_3\ v_4\ v_5\ v_6)$ where v_i denotes the i-th column.

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad v_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Recover the matrix A. (Be careful of the subscripts of column indices)

(1) $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in N(A) \iff Ax = 0 \iff \begin{cases} x_1 + 2x_2 - x_4 + 3x_6 = 0 \\ x_3 + x_4 + 2x_6 = 0 \\ x_5 + x_6 = 0 \end{cases}$ 1

$\iff \begin{cases} x_1 = -2x_2 + x_4 - 3x_6 \\ x_3 = -x_4 - 2x_6 \\ x_5 = -x_6 \end{cases}$ 3 x_2, x_4, x_6 are free variables.

$\Rightarrow N(A) = \mathrm{Span}\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ -2 \\ 0 \\ -1 \\ 1 \end{pmatrix} \right\}$ 3 3 3

(2) From the rref(A), we get the linear relation: 3

$v_2 = 2 \cdot v_1, \quad v_4 = -v_1 + v_3, \quad v_6 = 3v_1 + 2v_3 + v_5$

$\Rightarrow v_2 = 2 \cdot v_1 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$, 3 $v_4 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, 3

$v_5 = v_6 - 3v_1 - 2v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \\ 1 \end{pmatrix}$ 3

$$\boxed{A = \begin{pmatrix} 1 & 2 & 0 & -1 & -3 & 0 \\ 0 & 0 & 2 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}}$$

2(30pts) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be the linear transformation given by

$$T(f) = xf'(x) + f''(x) + f(0).$$

Let $\beta = \{1, x, x^2\}$ be the standard basis.

(1) Calculate the matrix representation $A = [T]_\beta$ w.r.t. the standard basis β.

(2) Find a matrix Q such that QAQ^{-1} is diagonal (**note:** not $Q^{-1}AQ$).

(1) $$[T]_\beta = \left([T(1)]_\beta \ \ [T(x)]_\beta \ \ [T(x^2)]_\beta \right) = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$[T(1)]_\beta = [0+0+1]_\beta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \ ③, \quad [T(x)]_\beta = [x\cdot 1+0+0]_\beta = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \ ③, \quad [T(x^2)]_\beta = [x\cdot 2x+2+0]_\beta = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix} \ ③$$

(2) Find eigenvalues: characteristic polynomial:

$$f(t) = \begin{vmatrix} 1-t & 0 & 2 \\ 0 & 1-t & 0 \\ 0 & 0 & 2-t \end{vmatrix} = (1-t)^2 \cdot (2-t) = 0 \Rightarrow \lambda = 1, 2 \quad ④$$

Find eigenvectors: $\lambda = 1$; $A - 1\cdot I = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ③+③

$\Rightarrow$ basis for $E_1 = N(A-I)$: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

$\lambda = 2$: $A - 2\cdot I = \begin{pmatrix} -1 & 0 & 2 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow v_3 = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ ③

$\Rightarrow P^{-1} A P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ with $P = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. ③

$\Rightarrow Q = P^{-1}$ calculation: $\Rightarrow Q = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{③\cdot(-2)+①} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

② ③

3(25pts) Let V be the subspace of $P_3(\mathbf{R})$ spanned by

$$1+x+x^2, \quad 2x+x^3, \quad 2+2x^2-x^3, \quad 1+x^3$$

(1) Find a basis for V.
(2) Is $f(x) = 2-x^2+x^3$ in the subspace V? Write down reason and calculations.

choose standard basis $\beta = \{1, x, x^2, x^3\}$. 4

$$A = \left([1+x+x^2]_\beta \; [2x+x^3]_\beta \; [2+2x^2-x^3]_\beta \; [1+x^3]_\beta \right) = \begin{pmatrix} 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 1 \end{pmatrix} = A.$$

Under the linear transformation $f(x) \mapsto [f(x)]_\beta$. $P_3(\mathbb{R})$ is isomorphic to $\mathbb{R}^4$.

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V is isomorphic to the column space of A.

$2-x^2+x^3$ is in $V \Leftrightarrow \underbrace{\begin{pmatrix} 2 \\ 0 \\ -1 \\ 1 \end{pmatrix}}_{=b}$ is in the column space of A.

Solve (1) and (2) together:

$$\underbrace{\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & -1 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right)}_{=(A|b)} \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 0 & 2 & -2 & -1 & -2 \\ 0 & 0 & 0 & -1 & -3 \\ 0 & 1 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -3 & -4 \end{array}\right)$$

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$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 2 \\ 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right)$$

6

$\Rightarrow$ leading 1's are in 1st, 3rd, 4-th column.

$\Rightarrow$ (1) basis for V: $\{1+x+x^2, 2+2x^2-x^3, 1+x^3\}$. 6

(2) $\text{rank}(A|b) = 4 > \text{rank}(A) = 3 \Rightarrow b$ is not in the column space of A

$\Rightarrow 2-x^2+x^3$ is not in the subspace V. 5

4(20pts) Let $S : V \to W$ and $T : W \to V$ be linear transformations between two vector spaces V and W. Let $T \circ S : V \to V$ be the composition.

(1) Prove that if S is surjective (i.e onto), then $\text{Rank}(T \circ S) = \text{Rank}(T)$.

(2) If S is injective (i.e. one-to-one), is $\text{Rank}(T \circ S) = \text{Rank}(T)$ always true? Prove this statement or find a counterexample.

(**Recall:** The rank of a linear transformation is the dimension of its range)

(1) Proof: Consider the range of linear transformations.

$$R(T\circ S) = T\circ S(V) = T(S(V)) = T(W) = R(T)$$ 6

S surjective $\Leftrightarrow$ $S(V) = W$ 4

In general

(2) $$R(T\circ S) = T\circ S(V) = T(S(V)) \subseteq T(W) = R(T)$$ 4

S injective $\not\Rightarrow$ $S(V) = W$ and $T(S(V)) \subseteq T(W)$ can be strict smaller

Example: $S: \mathbb{R} \to \mathbb{R}^2$ $S(x) = (x, 0)$

$T: \mathbb{R}^2 \to \mathbb{R}$ $T(x, y) = y$. 6

$$T\circ S(x) = T((x,0)) = 0 \Rightarrow \text{Rank}(T\circ S) = 0$$

$$\neq$$

$$\text{Rank}(T) = 1$$