!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : ID :

1(20pts) (i) Prove that $S = \{x^2+1, x^2+x\} \subset P_2(\mathbf{R})$ is linearly independent.
(ii) Extend S to a basis β of $P_2(\mathbf{R})$. Explain why your extension is a basis.

(i) We need to prove that there is no nontrivial representation of 0 as a linear combination of S.

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Assume $a\cdot(x^2+1) + b\cdot(x^2+x) = 0$.

$$a\cdot(x^2+1) + b\cdot(x^2+x) = (a+b)x^2 + bx + a$$

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Then $\begin{cases} a+b=0 \\ b=0 \\ a=0 \end{cases} \Rightarrow a=b=0$. so only trivial rep. of 0.

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(ii) $\beta = \{x^2+1, x^2+x, 1\}$ is a basis

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This is because that β is linearly independent and

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$\#\beta = 3 = \dim P_2(\mathbb{R})$.

$$a(x^2+1)+b(x^2+x)+c\cdot 1 = 0 \Rightarrow \begin{cases} a+b=0 \\ b=0 \\ a+c=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

$$a(x^2+1)+b(x^2+x)+c\cdot 1 = (a+b)x^2+b\cdot x+(a+c)$$

OR: β spans $P_2(\mathbb{R})$ and $\#\beta=3$:

$1=1$, $x = x^2+x-(x^2+1)+1$, $x^2 = (x^2+1)-1$.

2(20pts) Consider the linear transformation:

$$T : P_2(\mathbf{R}) \to M_{2\times 2}(\mathbf{R}),\quad T(f) = \begin{pmatrix} f(0) & f'(0) \\ f(1) & f'(1) \end{pmatrix}.$$

(1) Find the matrix representation of T with respect to the bases

$$\beta = \{1, x, x^2\},\quad \gamma = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

(2) Let $f(x) = 3 - 2x + x^2$. Calculate $[f(x)]_\beta$ and $[T(f(x))]_\gamma$.

(1) $f = a + bx + c\cdot x^2$ $\quad f(0) = a,\ f'(0) = b,\ f(1) = a+b+c,\ f'(1) = b+2c$ 1

$$\Rightarrow T(f) = \begin{pmatrix} a & b \\ a+b+c & b+2c \end{pmatrix}$$

$$[T]_\beta^\gamma = \left(\underset{\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}}{\underset{\parallel}{[T(1)]_\gamma}},\ \underset{\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}}{\underset{\parallel}{[T(x)]_\gamma}},\ \underset{\begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}}{\underset{\parallel}{[T(x^2)]_\gamma}} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$ 9

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(2) $$[f(x)]_\beta = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

$$[T(f(x))]_\gamma = [T]_\beta^\gamma \cdot [f(x)]_\beta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \\ 0 \end{pmatrix}.$$ 6

OR $$T(f(x)) = \begin{pmatrix} 3 & -2 \\ 2 & 0 \end{pmatrix} \Rightarrow [T(f(x))]_\gamma$$ //

3(20 pts) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined as:

$$T(x, y, z) = (y - 2z, x - 2z, x - y).$$

Calculate nulllity(T) and rank(T).

$$N(T) = \{ (x,y,z) \in \mathbb{R}^3 : T(x,y,z) = 0 \}$$

$$0 = T(x,y,z) = (y-2z, x-2z, x-y) \Leftrightarrow \begin{cases} y = 2z \\ x = 2z \\ x = y \end{cases}$$

$$\Updownarrow$$

$$\forall z \in \mathbb{R}.$$

$$(x,y,z) = (2z, 2z, z) = z \cdot (2,2,1)$$

So $N(T) = \text{span}\{(2,2,1)\}$. 8

nullity$(T) = \dim N(T) = 1$. 4

rank$(T) = \dim \mathbb{R}^3 - \text{nullity}(T) = 3 - 1 = 2$ 8

(1) Let $S \overset{\shortparallel}{=} \{v_1, \cdots, v_r\}$ be a linearly independent subset of V. Prove that $T(S) = \{T(v_1), \cdots, T(v_r)\}$ is a linearly independent subset of W.
(10 pts)

4(40pts) Let $T : V \to W$ be a linear transformation between finite dimensional vector spaces. Assume that T is one-to-one but **not** onto.

(2) (10pts) Prove the inequality $\dim V < \dim W$.

(3) (10pts) Prove that there exists a linear transformation $U : W \to V$ such that $UT = \mathrm{Id}_V$ (Hint: define U by its values on basis vectors).

(4) (10pts) Is there a linear transformation $U : W \to V$ such that $TU = \mathrm{Id}_W$? Explain your reasons.

Proof: (1) Assume $a_1 T(v_1) + a_2 T(v_2) + \cdots + a_r T(v_r) = 0$. 3

Then $T(a_1 v_1 + a_2 v_2 + \cdots + a_r v_r) = 0$ 3

Because T is one-to-one, $a_1 v_1 + a_2 v_2 + \cdots + a_r v_r = 0$. 3

Because $\{v_1, \cdots, v_r\}$ is linear independent, $a_1 = a_2 = \cdots = a_r = 0$. 1

So there is only trivial linear relation for $T(S)$. $T(S)$ is linearly independent.

(2). Choose a basis $\beta = \{v_1, v_2, \cdots, v_n\}$ for V. 2 β is linearly independent.

By part (1), $T(\beta) = \{T(v_1), T(v_2), \cdots, T(v_n)\}$ is linearly independent. 2

so $\#(T(\beta)) = \underset{\dim V}{\overset{\shortparallel}{n}} \leq \underset{\#\gamma}{\overset{\shortparallel}{\dim W}}$ because $T(\beta)$ can be extended to a basis γ for W. 3

Because $\dim V < \dim W$, $\gamma \neq T(\beta)$. Otherwise $\underset{T(V)}{\operatorname{Span}(T(\beta))} = \underset{W}{\operatorname{Span}(\gamma)}$ contradicting that T is not onto.

so. $\dim V = \#(T(\beta)) < \#\gamma = \dim W$. 2

Continuation of works:

(3) Continuing with the notation from part (2), assume:

$$\gamma = \{\underset{w_1}{\underset{\parallel}{T(v_1)}}, \underset{w_2}{\underset{\parallel}{T(v_2)}}, \cdots, \underset{w_n}{\underset{\parallel}{T(v_n)}}, w_{n+1}, \cdots, w_m\} \quad \text{with } n < m.$$

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Define the linear transformation U by setting:

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$$U(w_1) = v_1, \cdots, U(w_n) = v_n, \; U(w_{n+1}) = 0, \cdots, U(w_m) = 0.$$

Then $UT(v_i) = U(w_i) = v_i$, for $1 \leq i \leq n$.

So $UT = Id_V$ (because $\beta = \{v_1, \cdots, v_n\}$ is a basis)

(4). If $TU = Id_W$, then T is onto, which contradicts to the assumption that T is not onto. So there is no such U.

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