!!! WRITE YOUR NAME, STUDENT ID. BELOW !!!

NAME : ID :

1(25pts) A is a 3×5 matrix whose reduced row ethelon form is equal to:

$$\begin{pmatrix} 1 & 0 & -2 & 0 & -1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

(1) Find a basis for the null space $N(A)$.
(2) Assume that $A = (v_1\ v_2\ v_3\ v_4\ v_5)$ where v_i denotes the i-th column.

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Recover the matrix A. (Be careful of the subscripts of column indices)

(1) $x \in \mathbb{R}^5$ is in $N(A) \Leftrightarrow Ax = 0 \Leftrightarrow rref(A)x = 0$

$$\Leftrightarrow \begin{cases} x_1 - 2x_3 - x_5 = 0 \\ x_2 + x_3 + x_5 = 0 \\ x_4 + 2x_5 = 0 \end{cases} \Leftrightarrow \begin{cases} x_1 = 2x_3 + x_5 \\ x_2 = -x_3 - x_5 \\ x_4 = -2x_5 \end{cases}$$ 5

basis for $N(A)$ ↓

$$\Leftrightarrow x = \begin{pmatrix} 2x_3 + x_5 \\ -x_3 - x_5 \\ x_3 \\ -2x_5 \\ x_5 \end{pmatrix} = x_3 \cdot \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \in \mathrm{Span}\left\{ \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \\ -2 \\ 1 \end{pmatrix} \right\}$$

5 5

(2) From $rref(A)$, we get the linear relations:

$v_3 = -2v_1 + v_2 \Rightarrow v_2 = v_3 + 2 \cdot v_1 = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix}$ 5

$v_5 = -v_1 + v_2 + 2v_4 \Rightarrow v_5 = -\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ 5

$$\text{So } A = \begin{pmatrix} 1 & 2 & 0 & 0 & 1 \\ 1 & 4 & 2 & 1 & 5 \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix}$$

2(25pts) Let $T : P_2(\mathbf{R}) \to P_2(\mathbf{R})$ be the linear transformation given by

$$T(f) = x^2 f''(x) + f'(x) + 2f(x).$$

Let $\beta = \{1, x, x^2\}$ be the standard basis.

(1) Calculate the matrix representation $A = [T]_\beta$ w.r.t. the standard basis β.

(2) Determine whether T is diagonalizable or not. If yes, find Q such that $Q^{-1}AQ$ is a diagonal matrix.

(1) $A = \left([T(1)]_\beta \; [T(x)]_\beta \; [T(x^2)]_\beta \right) = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 4 \end{pmatrix}$

$[T(1)]_\beta = [2]_\beta$, $[T(x)]_\beta = [1+2x]_\beta$, $[T(x^2)]_\beta = [x^2 \cdot 2 + 2x + 2x^2]_\beta = [2x + 4x^2]_\beta$

5 5

(2) Find eigenvalues of T (or equivalently for A).

$$A - \lambda I = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 2 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (2-\lambda)^2 \cdot (4-\lambda) = 0 \Rightarrow \begin{array}{l} \lambda = 2, \; m_2 = 2 \\ \lambda = 4, \; m_4 = 1 \end{array}$$

multiplicity

5

Calculate $\dim E_2 = \dim N(A - 2I)$:

$$A - 2I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \dim E_2 = \#\text{free var.} = 1$$

5

$1 = \dim E_2 < m_2 = 2 \Rightarrow A$ is not diagonalizable

$\Leftrightarrow T$ is not diagonalizable

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3(25pts) Let V be the subspace of $P_3(\mathbf{R})$ spanned by

$$1+x+x^2,\quad x+2x^2-x^3,\quad 1-x^2+x^3,\quad 2+x+x^3$$

(1) Find a basis β for V.
(2) Extend β to a basis γ of $P_3(\mathbb{R})$. Explain why the subset γ you obtain is a basis for $P_3(\mathbb{R})$.

(1) Use standard basis $\alpha = \{1, x, x^2, x^3\}$

$$\left([v_1]_\alpha\ [v_2]_\alpha\ [v_3]_\alpha\ [v_4]_\alpha\right) = \begin{pmatrix} 1 & 0 & 1 & 2 \\ 1 & 1 & 0 & 1 \\ 1 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{pmatrix} = A \quad 5$$

$$\to \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & -2 & -2 \\ 0 & -1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ -1 \end{pmatrix} \right\} \text{ is a basis}$$

$\uparrow\ \uparrow$ leading 1's 5

for the column space of A 5

$\Rightarrow \{v_1, v_2\} = \{1+x+x^2,\ x+2x^2-x^3\}$ is a basis for V

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(2) $\beta = \{1+x+x^2,\ x+2x^2-x^3,\ 1,\ x\}$ is a basis for $P_3(\mathbb{R})$

because: $1 = 1\cdot 1$, $x = 1\cdot x \in \operatorname{Span}\beta$

$x^2 = 1\cdot(1+x+x^2) - 1\cdot 1 - 1\cdot x \in \operatorname{Span}\beta$

$x^3 = -1\cdot(x+2x^2-x^3) + 1\cdot x + 2\cdot x^2$

Span β

$= -1(x+2x^2-x^3) + 1\cdot x + 2\cdot(1+x+x^2) - 2\cdot 1 - 2\cdot x$

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$\Rightarrow P_3(\mathbb{R}) = \operatorname{Span}\beta$ and $\#\beta = 4 = \dim P_3(\mathbb{R})$

4(25pts) Consider two 5×5 blocked matrices:

$$S = \left(\begin{array}{c|c} A & B \\ \hline \mathbf{0} & C \end{array}\right), \quad T = \left(\begin{array}{c|c} \mathbf{0} & C \\ \hline A & B \end{array}\right)$$

where A is a 2×2 matrix, B is a 2×3 matrix and C is a 3×3 matrix.

(i) Prove that $\det(S) = \det(A)\det(C)$ (Explain your idea and argument).

(ii) If $\det(T) = k \cdot \det(S)$ for a constant $k \in \mathbf{R}$. Determine the constant k.

(iii) Calculate $\det(S)$ and $\det(T)$ when

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

(i) transform A to $\mathrm{rref}(A)$: $\mathrm{rref}(A) = E_p \cdots E_1 A$ E_i are 2×2 elementary matrices.

C to $\mathrm{rref}(C)$: $\mathrm{rref}(C) = F_q \cdots F_1 \cdot C$ F_j are 3×3 elementary matrices.

$$\Rightarrow \left(\begin{array}{c|c} E_p & 0 \\ \hline 0 & I_3 \end{array}\right) \cdots \left(\begin{array}{c|c} E_1 & 0 \\ \hline 0 & I_3 \end{array}\right) \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & F_q \end{array}\right) \cdots \left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & F_1 \end{array}\right) S = \underbrace{\left(\begin{array}{c|c} \mathrm{rref}(A) & B' \\ \hline 0 & \mathrm{rref}(C) \end{array}\right)}_{\text{upper triangular}}$$

$$\Rightarrow \det\left(\begin{array}{c|c} E_p & 0 \\ \hline 0 & I_3 \end{array}\right) \cdots \det\left(\begin{array}{c|c} E_1 & 0 \\ \hline 0 & I_3 \end{array}\right) \det\left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & F_q \end{array}\right) \cdots \det\left(\begin{array}{c|c} I_2 & 0 \\ \hline 0 & F_1 \end{array}\right) \det S = \det(\mathrm{rref}(A)) \det(\mathrm{rref}(C))$$

$$\Rightarrow \det(E_p) \cdots \det(E_1) \cdot \det(F_q) \cdots \det(F_1) \cdot \det(S) = \det(\mathrm{rref}(A)) \cdot \det(\mathrm{rref}\, C).$$

$$\Rightarrow \det(S) = \det(E_p)^{-1} \cdots \det(E_1)^{-1} \det(\mathrm{rref}(A)) \cdot \det(F_q)^{-1} \cdots \det(F_1)^{-1} \cdot \det(\mathrm{rref}\, C)$$

$$= \det(A) \cdot \det(C)$$

Another proof: two cases: Case 1: $\det(A) = 0 \Leftrightarrow \mathrm{rank}(A) < 2 \Rightarrow \mathrm{rank}(S) < 5)$

$\Rightarrow \det(S) = 0$

Case 2: $\det(A) \neq 0 \Rightarrow$ $\left(\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & I_3 \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) = \left(\begin{array}{c|c} I_2 & A^{-1}B \\ \hline 0 & C \end{array}\right)$

5 + 5

$\Rightarrow \det(A)^{-1} \cdot \det S = \det C \Rightarrow \det(S) = \det A \cdot \det C$

There are other proofs (by using induction for example)

(ii) T can be obtained from S by switching rows

$$\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} \to \begin{pmatrix} r_4 \\ r_2 \\ r_3 \\ r_1 \\ r_5 \end{pmatrix} \to \begin{pmatrix} r_4 \\ r_5 \\ r_3 \\ r_1 \\ r_2 \end{pmatrix} \to \begin{pmatrix} r_3 \\ r_5 \\ r_4 \\ r_1 \\ r_2 \end{pmatrix} \to \begin{pmatrix} r_3 \\ r_4 \\ r_5 \\ r_1 \\ r_2 \end{pmatrix}$$ 5

$\Rightarrow \det(T) = (-1)^4 \cdot \det(S) = 1 \cdot \det(S)$. So $k=1$. 5

(iii) $\det(A) = 0 \cdot 0 - (-1) \cdot 1 = 1$. $\det(C) = 1 \cdot \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$ 5

$\Rightarrow \det(S) = 1 = \det(T)$. $\cancel{5}$