HW9- solution

4.2.3. $\det\begin{pmatrix} 2r_1 \\ 3r_2+5r_3 \\ 7r_3 \end{pmatrix} = \det\begin{pmatrix} 2r_1 \\ 3r_2 \\ 7r_3 \end{pmatrix} + \det\begin{pmatrix} 2r_1 \\ 5r_3 \\ 7r_3 \end{pmatrix}$

$= 42\det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + 35\cdot\det\begin{pmatrix} 2r_1 \\ r_3 \\ r_3 \end{pmatrix}$

$= 42\det\begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix} + 0$

$\Rightarrow k = 42.$

21. Can use elementary row operation of type 3 to transform into an upper triangular matrix:

$$\begin{vmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 0 & 3 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 19 & -38 \end{vmatrix}$$

$$95 = 1\cdot 1\cdot 19\cdot 5 = \begin{vmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 0 & 5 \end{vmatrix}$$

Can also use elementary column operations:

$$\begin{vmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 4 & -5 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 \\ 0 & 4 & 19 & -43 \\ 2 & 3 & 19 & -38 \end{vmatrix}$$

|| expand along the 1st. column

$$95 = 19\cdot 5 = 19\cdot(-38) + 19\cdot 43 = 1\cdot 1\cdot \begin{vmatrix} 19 & -43 \\ 19 & -38 \end{vmatrix}$$

26. $A \in M_{n\times n}(F)$

$\det(-A) = (-1)^n \cdot \det(A)$

$\det(-A) = \det A \iff (-1)^n = 1$.

If we assume $F = \mathbb{R}$ ($\operatorname{char} F \neq 2$), then $(-1)^n = 1 \iff n$ is even.

(If $\operatorname{char}(F) = 2$, e.g. $F = \mathbb{Z}_2$, then $\det(-A) = \det(A)$ always true).

4.3.11 $M \in M_{n\times n}(\underset{\text{complex numbers.}}{\mathbb{C}})$ skew symmetric : $M^t = -M$

$\det(M) = \det(M^t) = \det(-M) = (-1)^n \det(M)$.

if n is odd, then $\det(M) = -\det(M) \Rightarrow \det(M) = 0$.

When n is even, $\det(M)$ can be nonzero: for example $M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

20. $M = \begin{pmatrix} A & B \\ \mathbf{0} & I \end{pmatrix}$ where A is a square matrix. Prove $\det(M) = \det(A)$

__Proof__: Use induction on the size of $I = I_{k\times k}$

If $k=1$, then $M = \begin{pmatrix} A & B \\ \mathbf{0} & 1 \end{pmatrix} = \begin{pmatrix} A & \begin{matrix} b_1 \\ \vdots \\ b_{n-1} \end{matrix} \\ 0 \cdots 0 & 1 \end{pmatrix}$

Expanding along the last column, we get

$$\det(M) = (-1)^{n+n} \cdot 1 \cdot \det(A) - (-1)^{n-1+n} \cdot b_{n-1} \cdot \det\begin{pmatrix} * \\ 0 \cdots 0 \end{pmatrix} + \cdots (-1)^{1+n} \begin{vmatrix} * \\ 0 \cdots 0 \end{vmatrix}$$

$$= \det(A).$$

Assume case k is true.

Expanding along the last column, we get

$$\underset{\det(M)}{\overset{\|}{|M|}} = \begin{vmatrix} A & B \\ 0 & I_{k+1} \end{vmatrix} = \begin{vmatrix} A & B' & \overset{\text{last column}}{\downarrow} \\ & & v \\ 0 \cdots 0 & \begin{matrix} 1 & & \\ & \ddots & \\ & & 1 \end{matrix} & \begin{matrix} 0 \\ \vdots \end{matrix} \\ 0 \cdots 0 & 0 \cdots 0 & 1 \end{vmatrix} \quad \text{where } B = (B' \vdots v)$$

$$= 1 \cdot \begin{vmatrix} A & B' \\ 0 & I_k \end{vmatrix} = 1 \cdot \det(A) \text{ by induction}$$

$\blacksquare$

(a) implies (b). Just need to prove:

23. Let k denote the largest integer s.t. some $k \times k$ submatrix has a nonzero determinant. Prove that $\operatorname{rank}(A) = k$.

<u>Proof</u>: Recall: $\operatorname{rank}(A) = k \iff \dim(\underbrace{\text{column space of } A}_{= V}) = k = \dim(\underbrace{\text{row space of } A}_{= W})$

$\iff \exists\ k$ linearly independent column vectors $v_{i_1}, \dots, v_{i_k}$ that span V

$\iff \exists\ k$ linearly independent row vectors $r_{j_1}, \dots, r_{j_k}$ that span W

We will prove

(1) $\exists$ $(m \times m)$-submatrix C s.t. $\det(C) \neq 0 \Rightarrow \operatorname{rank}(A) \geq m$

(2) $\operatorname{rank}(A) = k \Rightarrow \exists$ $(k \times k)$-submatrix D, s.t. $\det(D) \neq 0$.

These will imply the wanted statement.

Proof of (1): Let $C = (v_{i_1}' \cdots v_{i_m}')$ be a submatrix of A such that $\det(C) \neq 0$. Then $v_{i_1}', \cdots, v_{i_m}'$ are linearly independent

$\Rightarrow$ the corresponding columns $v_{i_1}, \cdots, v_{i_m}$ of A are linearly independent

$\Rightarrow$ $\operatorname{rank}(A) \geq m$.

Proof of (2): Assume $\operatorname{rank}(A) = k$.

Let $v_{i_1}, \cdots, v_{i_k}$ be linearly independent columns of A

Set $B = (v_{i_1} \cdots v_{i_k})$. Then B is an $n \times k$ matrix with $\operatorname{rank}(B) = k$:

$\operatorname{rank}(B) = k \Rightarrow \dim(\text{row space of } B) = k$

$\Rightarrow \exists$ k linearly independent rows $r_{j_1}', \cdots, r_{j_k}'$ of B

Set $D = \begin{pmatrix} r_{j_1}' \\ \vdots \\ r_{j_k}' \end{pmatrix}$. Then $\operatorname{rank}(D) = k \Rightarrow \det(D) \neq 0$.

24.

$$|A+tI| = \begin{vmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1}+t \end{vmatrix}$$

expand along the 1st row

$$= t \cdot \begin{vmatrix} t & 0 & \cdots & 0 & a_1 \\ -1 & t & \cdots & 0 & a_2 \\ & & & & \\ 0 & \cdots & & -1 & a_{n-1}+t \end{vmatrix} + \underbrace{a_0 \cdot (-1)^{1+n} \cdot \begin{vmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{vmatrix}}_{= a_0 \cdot (-1)^{1+n} \cdot (-1)^{n-1} = a_0}$$

iteration / induction

$$= t \cdot \left(t \cdot \begin{vmatrix} t & 0 & \cdots & 0 & a_2 \\ -1 & t & \cdots & 0 & a_3 \\ & & \cdots & & \\ 0 & \cdots & & -1 & a_{n-1}+t \end{vmatrix} + a_1 \right) + a_0$$

$$= t^2 \cdot \begin{vmatrix} t & 0 & \cdots & 0 & a_2 \\ -1 & t & \cdots & 0 & a_3 \\ & & \cdots & & \\ 0 & \cdots & & -1 & a_{n-1}+t \end{vmatrix} + t \cdot a_1 + a_0$$

$$= \cdots$$

$$= t^{n-2} \cdot \begin{vmatrix} t & a_{n-2} \\ -1 & a_{n-1}+t \end{vmatrix} + t^{n-3} a_{n-3} + \cdots + t \cdot a_1 + a_0$$

$$= t^{n-2} \cdot (t^2 + a_{n-1} t + a_{n-2}) + t^{n-3} a_{n-3} + \cdots + t a_1 + a_0$$

$$= t^n + t^{n-1} a_{n-1} + t^{n-2} a_{n-2} + t^{n-3} a_{n-3} + \cdots + t \cdot a_1 + a_0$$

Another way:

$$|A+tI| = \begin{vmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -1 & a_{n-1}+t \end{vmatrix}$$

$$= \begin{vmatrix} t & 0 & 0 & \cdots & 0 & a_0 \\ -1 & t & 0 & \cdots & 0 & a_1 \\ 0 & -1 & t & \cdots & 0 & a_2 \\ \vdots & \vdots & \vdots & t & \vdots & \vdots \\ 0 & 0 & 0 & \cdots -1 & 0 & (a_{n-1}+t)\cdot t + a_{n-2} \\ 0 & 0 & 0 & \cdots 0 & -1 & a_{n-1}+t \end{vmatrix}$$

$$= \begin{vmatrix} t & 0 & 0 & \cdots & & 0 & a_0 \\ -1 & t & 0 & \cdots & & 0 & a_1 \\ & & \cdots & & & & \cdots \\ 0 & 0 & 0 & \cdots -1 & 0 & 0 & t(a_{n-1}t+t^2+a_{n-2})+a_{n-3} \\ 0 & 0 & 0 & \cdots & -1 & 0 & a_{n-1}t+t^2+a_{n-2} \\ 0 & 0 & 0 & \cdots & 0 & -1 & a_{n-1}+t \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & t^n+a_{n-1}t^{n-1}+a_{n-2}t^{n-2}+\cdots+a_1t+a_0 \\ -1 & 0 & 0 & \cdots & 0 & 0 & \vdots \\ & & & & & & \\ 0 & 0 & 0 & \cdots & 0 & -1 & a_{n-1}+t \end{vmatrix}$$

$$= (-1)^{1+n} \begin{vmatrix} -1 & & \\ & \ddots & \\ & & -1 \end{vmatrix}_{(n-1)\times(n-1)} \cdot \left(t^n+a_{n-1}t^{n-1}+a_{n-2}t^{n-2}+\cdots+a_1t+a_0\right)$$

$$= (-1)^{1+n}(-1)^{n-1}\cdot\left(t^n+a_{n-1}t^{n-1}+a_{n-2}t^{n-2}+\cdots+a_1t+a_0\right)$$

$$= t^n+a_{n-1}t^{n-1}+a_{n-2}t^{n-2}+\cdots+a_1t+a_0.$$