3.3.8: $T: \mathbb{R}^3 \to \mathbb{R}^3$ $\quad T(a,b,c) = (a+b,\ b-2c,\ a+2c)$

Determine whether $v \in R(T)$

(a) $v = (1, 3, -2)$

Determine whether $\begin{cases} a+b=1 \\ b-2c=3 \\ a+2c=-2 \end{cases}$ is solvable:

Augmented matrix: $\left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 1 & 0 & 2 & -2 \end{array}\right) \xrightarrow{①\cdot(-1)+③} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & -1 & 2 & -3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \operatorname{rank}(A|b) = \operatorname{rank}(A) = 2 \Rightarrow$ solvable $\Rightarrow v \in R(T)$.

(b) $v = (2\ 1\ 1)$

Same method $\rightsquigarrow$ augmented matrix:

$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 1 & 0 & 2 & 1 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & -1 & 2 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$\Rightarrow \operatorname{rank}(A|b) = \operatorname{rank}(A) = 2 \Rightarrow v \in R(T)$.

10: coefficient matrix A: $m \times n$, $\operatorname{rank}(A) = m \Rightarrow Ax=b$ has a solvable.

This is true. Proof: Consider $L_A: \mathbb{R}^n \to \mathbb{R}^m$, $L_A(v) = Av$

$\operatorname{rank}(A) = m \Rightarrow \operatorname{rank}(L_A) = \dim(R(L_A)) = m \Rightarrow R(L_A) = \mathbb{R}^m$

$\Rightarrow L_A$ is surjective $\Rightarrow Ax=b$ has a solution for any $b \in \mathbb{R}^m$.

3.4.3: $(A|b) \longrightarrow$ reduced row echelon form $(A'|b')$

(a) Prove $\text{rank}(A') \neq \text{rank}(A'|b') \Leftrightarrow (A'|b')$ contains a row in which the only nonzero entry lies in the last column.

Proof: $\text{rank}(A') = \text{rank}(A'|b') \Leftrightarrow$ column space of A' $=$ column space of $(A'|b')$

$\Leftrightarrow b' \in$ column space of A'.

If A' has r nonzero rows, then A' has r leading ones.

The column space of A' is equal to $\left\{ \begin{pmatrix} a_1 \\ \vdots \\ a_r \\ 0 \\ \vdots \\ 0 \end{pmatrix} ; a_1, \dots, a_r \in \mathbb{R} \right\}$.

So $b' \in$ Column space of $A' \Leftrightarrow$ if a row of A' is 0, then the last entry in the same row of $(A'|b')$ is also equal to 0.

(b) $Ax = b$ is consistent $\Leftrightarrow A'x = b'$ is consistent

because elementary row operations transform a linear system into an equivalent linear system.

By (a), this is equivalent to the condition that $(A'|b')$ contains no row in which the only nonzero entry lies in the last column.

4.(a) $\begin{cases} x_1+2x_2-x_3+x_4=2 \\ 2x_1+x_2+x_3-x_4=3 \\ x_1+2x_2-3x_3+2x_4=2 \end{cases}$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 2 & 1 & 1 & -1 & 3 \\ 1 & 2 & -3 & 2 & 2 \end{array}\right) \xrightarrow[\text{①}\cdot(-1)+\text{③}]{\text{①}\cdot(-2)+\text{②}} \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & -3 & 3 & -3 & -1 \\ 0 & 0 & -2 & 1 & 0 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 0 & \frac{1}{2} & 2 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right) \longleftarrow \left(\begin{array}{cccc|c} 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & -1 & 1 & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right)$$

$$\downarrow \text{②}\cdot(-2)+\text{①}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & -\frac{1}{2} & \frac{4}{3} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{3} \\ 0 & 0 & 1 & -\frac{1}{2} & 0 \end{array}\right) \Longrightarrow \begin{cases} x_1 = \frac{4}{3} + \frac{1}{2}x_4 \\ x_2 = \frac{1}{3} - \frac{1}{2}x_4 \\ x_3 = \frac{1}{2}x_4 \end{cases}$$

basis for solution set of homogeneous system: $\beta = \left\{ \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix} \right\}$

set $x_4 = 1$.

5. reduced row echelon form of A:

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}$$

1st. 2nd and 4-th column of A:

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}.$$

Determine A: Elementary row operations do not change linear relations between columns. From $rref(A)$:

$$v_3' = 2v_1' - 5v_2' \Rightarrow v_3 = 2v_1 - 5v_2 = 2\cdot\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 5\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$$

$$v_5' = -2v_1' - 3v_2' + 6v_4' \Rightarrow v_5 = -2v_1 - 3v_2 + 6v_4$$

$$= -2\cdot\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} - 3\cdot\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} + 6\cdot\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} = \begin{pmatrix} 4 \\ -7 \\ -9 \end{pmatrix}$$

$$\Rightarrow A = (v_1\ v_2\ v_3\ v_4\ v_5) = \begin{pmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{pmatrix}$$

7. Use u_i as column vectors to form a matrix:

$$\begin{pmatrix} 2 & 1 & -8 & 1 & -3 \\ -3 & 4 & 12 & 37 & -5 \\ 1 & -2 & -4 & -17 & 8 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 5 & 0 & 35 & -19 \\ 0 & -2 & 0 & -14 & 19 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 1 & 0 & 7 & -\frac{19}{2} \\ 0 & 1 & 0 & 7 & -\frac{19}{5} \end{pmatrix}$$

$$\leftarrow$$

$$\begin{array}{c} \begin{pmatrix} 1 & -2 & -4 & -17 & 8 \\ 0 & 1 & 0 & 7 & -\frac{19}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \\ \begin{array}{ccccc} \uparrow & \uparrow & & & \uparrow \\ \text{1st.} & \text{2nd} & & & \text{5-th} \end{array} \end{array}$$

$$\Rightarrow \text{basis} := \{u_1, u_2, u_5\} = \left\{ \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ 8 \end{pmatrix} \right\}$$

8. Form a matrix by u_i:

$$\begin{pmatrix} 2 & -6 & 3 & 2 & -1 & 0 & 1 & 2 \\ -3 & 9 & -2 & -8 & 1 & -3 & 0 & -1 \\ 4 & -12 & 7 & 2 & 2 & -18 & -2 & 1 \\ -5 & 15 & -9 & -2 & 1 & 9 & 3 & -9 \\ 2 & -6 & 1 & 6 & -3 & 12 & -2 & 7 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & \frac{3}{2} & 1 & -\frac{1}{2} & 0 & \frac{1}{2} & 1 \\ 1 & -3 & \frac{2}{3} & \frac{8}{3} & -\frac{1}{3} & 1 & 0 & \frac{1}{3} \\ 1 & -3 & \frac{7}{4} & \frac{1}{2} & \frac{1}{2} & -\frac{9}{2} & -\frac{1}{2} & \frac{1}{4} \\ 1 & -3 & \frac{9}{5} & \frac{2}{5} & -\frac{1}{5} & \frac{9}{5} & -\frac{3}{5} & \frac{9}{5} \\ 1 & -3 & \frac{1}{2} & 3 & -\frac{3}{2} & 6 & -1 & \frac{7}{2} \end{pmatrix}$$

$$\Downarrow$$

$$\underline{\text{rref}} := \begin{array}{c} \begin{pmatrix} 1 & -3 & 0 & 4 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -4 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{cccccccc} \uparrow & & \uparrow & & \uparrow & & \uparrow & \\ \text{1st} & & \text{3rd} & & \text{5-th} & & \text{7-th} & \end{array} \end{array}$$

basis:
$$\left\{ \begin{pmatrix} 2 \\ -3 \\ 4 \\ -5 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 7 \\ -9 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 3 \\ -2 \end{pmatrix} \right\}$$

4.1.11: $\delta: M_{2\times2}(F) \to F$ satisfies:

(i) δ is a linear function of each row when the other row is held fixed

(ii) If two rows of A are identical, then $\delta(A)=0$

(iii) $\delta(I_{2\times2})=1$.

(a) Prove $\delta(E)=\det(E)$ for all elementary matrices $E \in M_{2\times2}(F)$.

- $E = E^{①\leftrightarrow②} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$0 \overset{(ii)}{=} \delta\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \overset{(i)}{=} \delta\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$= 0 + 1 + \delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + 0$$

So $\boxed{\delta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = -1.}$

- $E = E^{c\cdot①} = \begin{pmatrix} c & 0 \\ 0 & 1 \end{pmatrix}$ By (i) $\boxed{\delta(E) = c\cdot\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = c.}$

similarly for $E = E^{c\cdot②} = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix}$ $\boxed{\delta(E) = c\cdot\delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = c.}$

- $E = E^{c①+②} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}$

$$\delta(E) = \delta\begin{pmatrix} 1 & 0 \\ c & 0 \end{pmatrix} + \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = c \cdot \delta\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} + 1 = c0+1 = 1$$

Similarly for $E = E^{c②+①} = \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$

$$\delta(E) = \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & c \\ 0 & 1 \end{pmatrix} = 1 + c \cdot \delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 1 + c0 = 1.$$

(b) Prove $\delta(EA) = \delta(E) \cdot \delta(A)$

- $E = E^{①\leftrightarrow②}$ $\quad A = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ $\quad EA = \begin{pmatrix} r_2 \\ r_1 \end{pmatrix}$.

$$0 = \delta\begin{pmatrix} r_1+r_2 \\ r_1+r_2 \end{pmatrix} = \delta\begin{pmatrix} r_1 \\ r_1+r_2 \end{pmatrix} + \delta\begin{pmatrix} r_2 \\ r_1+r_2 \end{pmatrix}$$

$$= \delta\begin{pmatrix} r_1 \\ r_1 \end{pmatrix} + \delta\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} + \delta\begin{pmatrix} r_2 \\ r_1 \end{pmatrix} + \delta\begin{pmatrix} r_2 \\ r_2 \end{pmatrix}$$

$$= 0 + \delta(A) + \delta(EA) + 0$$

$$\Rightarrow \delta(EA) = -\delta(A) = \delta(E) \cdot \delta(A).$$

- $E^{c\cdot①}$ $\quad A = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$ $\quad EA = \begin{pmatrix} c \cdot r_1 \\ r_2 \end{pmatrix}$

$$\delta(EA) = \delta\begin{pmatrix} cr_1 \\ r_2 \end{pmatrix} = c\delta\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = c\delta(A) = \delta(E)\,\delta(A)$$

Similarly for $E^{c\cdot②}$

$\cdot\ E^{c\cdot① + ②}$. $EA = \begin{pmatrix} r_1 \\ c\cdot r_1 + r_2 \end{pmatrix}$

$$\delta(EA) = \delta\begin{pmatrix} r_1 \\ cr_1 + r_2 \end{pmatrix} = \delta\begin{pmatrix} r_1 \\ cr_1 \end{pmatrix} + \delta\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = c\cdot\delta\begin{pmatrix} r_1 \\ r_1 \end{pmatrix} + \delta\begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$$

$$= c\cdot 0 + \delta(A) = \delta(A) = \delta(E)\cdot\delta(A).$$ ☒

12. $\delta: M_{2\times 2}(F) \to F$ from 11.

Prove $\delta(A) = \det(A)$.

Proof: Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$.

Case 1: $a_{11} \neq 0$. Let $E_1 = E^{-a_{11}^{-1}\cdot a_{21}\cdot ① + ②}$. Then

$$E_1\cdot A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & -a_{11}^{-1}a_{21}\cdot a_{12} + a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & \underbrace{a_{11}^{-1}\cdot \det A}_{= b} \end{pmatrix}$$

Case 1(a): If $\det(A) = 0$, then $E_1\cdot A$ has a zero row.

Then $\delta(E_1\cdot A) = 0 = \delta(E_1)\cdot\delta(A) \Rightarrow \delta(A) = 0 = \det A = 0$.

Case 1(a): If $\det(A) \neq 0$, then let $E_2 = E^{-b^{-1}② + ①}$. Then

$$E_2\cdot E_1\cdot A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{11}^{-1}\det(A) \end{pmatrix}.$$

Then $\delta(E_2 E_1 A) = a_{11} \cdot a_{11}^{-1} \det(A) \cdot \delta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{(iii)}{=} \det(A).$

$$\begin{gathered} \delta(E_2 E_1 A) \\ \| \\ \delta(E_2) \cdot \delta(E_1) \cdot \delta(A) = \delta(A) \quad \checkmark \end{gathered}$$

<u>Case 2</u> : $a_{11} = 0$,

<u>Case 2a</u> : $a_{21} \neq 0$, then let $E = E^{① \leftrightarrow ②}$

$$E \cdot A = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix} \text{ satisfies case 1.}$$

So $\delta(EA) = \det(E \cdot A) = a_{21} \cdot a_{12} - a_{22} \cdot a_{11} = -\det(A)$

$$\begin{gathered} \delta(EA) \\ \| \\ \delta(E) \cdot \delta(A) = -1 \cdot \delta(A). \end{gathered}$$

So $\delta(A) = \det(A)$.

<u>Case 2b</u>: $a_{21} = 0$, then $A = \begin{pmatrix} 0 & a_{12} \\ 0 & a_{22} \end{pmatrix}$

$$\Rightarrow \delta(A) \overset{(i)}{=} a_{12} \cdot \delta\begin{pmatrix} 0 & 1 \\ 0 & a_{22} \end{pmatrix} \overset{(i)}{=} a_{12} \cdot a_{22}\, \delta\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \overset{(ii)}{=} 0$$