HW 7 Solutions

3.1.4: Proof:

Type 1: $E^{i \leftrightarrow j}$ interchanging i-th and j-th row

$\qquad\;\; \parallel$

$\qquad E^{\boxed{i} \leftrightarrow \boxed{j}}$ interchange i-th and j-th column.

$$\begin{array}{c|c} & \begin{array}{ccc} & i & j \end{array} \\ \begin{array}{c} i \\ j \end{array} & \begin{pmatrix} \cdots & 0 & \cdots & 1 \\ \cdots & 1 & \cdots & 0 \end{pmatrix} \end{array}$$

Type 2: $E^{c\cdot i}$ multiply i-th row by $c \neq 0$

$\qquad\;\; \parallel$

$\qquad E^{c\cdot\boxed{i}}$ multiply i-th column by $c \neq 0$.

$$\begin{pmatrix} \ddots & & & \\ & 1 & & \\ & & c & \\ & & & 1 \\ & & & & \ddots \end{pmatrix}$$

Type 3: $E^{c\,i + j}$ add the c-multiple of i-th row to the j-th row

$\qquad\;\; \parallel$

$\qquad E^{c\boxed{j} + \boxed{i}}$ add the c-multiple of j-th column to the i-th column.

$$\begin{array}{c} i \\ j \end{array}\begin{pmatrix} \cdots 1 & 0 \\ \cdots 0 & 1 \end{pmatrix} \xrightarrow{c\,i + j} \begin{array}{c} i \\ j \end{array}\begin{pmatrix} 1 & \cdots & 0 \\ c & \cdots & 1 \end{pmatrix}_{\;i \quad j} \xleftarrow{c\boxed{j}+\boxed{i}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\;i \quad j}$$

∎

6. Proof: B obtained from A by an elementary row operation

$\Rightarrow$ $B = E\cdot A$ with E a corresponding elementary matrix

$\Rightarrow$ $B^t = A^t \cdot E^t$ with E^t also an elementary row operation.

type 1: $(E^{i \leftrightarrow j})^t = E^{\boxed{i} \leftrightarrow \boxed{j}}$

type 2: $(E^{c\cdot i})^t = E^{c\cdot \boxed{i}}$

type 3: $(E^{c\,i + j})^t = E^{c\cdot\boxed{j} + \boxed{i}}$

$\Rightarrow$ B^t can be obtained from A^t by the corresponding elementary column operation ∎

3.2 5 (d)

$$\left(\begin{array}{ccc|ccc} 0 & -2 & 4 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 1 & 0 \\ 2 & 4 & -5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{②}\leftrightarrow\text{①}]{\text{②}\cdot(-2)+\text{③}} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 0 & 0 \\ 0 & 2 & -3 & 0 & -2 & 1 \end{array}\right)$$

$$\xrightarrow[\text{②}/\cdot(-2)]{\text{②}+\text{③}} \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & -2 & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \xrightarrow[\text{③}\cdot 1+\text{①}]{\text{③}\cdot 2+\text{②}} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & -1 & 1 \\ 0 & 1 & 0 & \frac{3}{2} & -4 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right)$$

$$\xrightarrow{\text{②}\cdot(-1)+\text{①}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2} & 3 & -1 \\ 0 & 1 & 0 & \frac{3}{2} & -4 & 2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right) \Rightarrow \boxed{\begin{array}{l} \operatorname{rank} A = 4, \text{ invertible} \\ A^{-1} = \begin{pmatrix} -\frac{1}{2} & 3 & -1 \\ \frac{3}{2} & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix} \end{array}}$$

Check: $\begin{pmatrix} 0 & -2 & 4 \\ 1 & 1 & -1 \\ 2 & 4 & -5 \end{pmatrix}\begin{pmatrix} -\frac{1}{2} & 3 & -1 \\ \frac{3}{2} & -4 & 2 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ✓

(h)

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & -1 & 2 & 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[\text{①}\cdot(-2)+\text{③}]{\text{①}\cdot(-1)+\text{②}} \left(\begin{array}{cccc|cccc} 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & -3 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\text{③}\cdot(-1)]{\text{②}+\text{④}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \\ 0 & 1 & -2 & 1 & \\ 0 & 0 & +1 & +2 & * \\ 0 & 0 & -1 & -2 & \end{array}\right) \xrightarrow{\text{③}+\text{④}} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 1 & \\ 0 & 1 & -2 & 1 & \\ 0 & 0 & 1 & 2 & * \\ 0 & 0 & 0 & 0 & \end{array}\right)$$

$\Rightarrow \operatorname{rank}(A) = 3 \neq 4$, not invertible.

9. Proof: an elementary column operation on A

$$\Updownarrow$$

$A \rightsquigarrow A\cdot E$ for a corresponding elementary matrix.

E invertible $\Rightarrow$ $\text{rank}(A) = \text{rank}(A\cdot E)$ by Thm 3.4 $\blacksquare$

15: A, B are matrices with n rows.

M : $m\times n$ matrix.

Assume $A = (v_1 \cdots v_k)$, $B = (u_1 \cdots u_l)$. $v_i \in \mathbb{R}^n,\ i = 1,\dots,k$; $u_j \in \mathbb{R}^n,\ j = 1,\dots,l$.

$$\begin{aligned} M\cdot(A|B) &= M(v_1 \cdots v_k \mid u_1 \cdots u_l) \\ &= (Mv_1 \cdots Mv_k \mid Mu_1 \cdots Mu_l) \\ &= (MA \mid MB) \end{aligned}$$

We used: if $M = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}$ with $r_p \in \mathbb{R}^n$, $p = 1,\dots,m$.

then $MA = \begin{pmatrix} r_1 \\ \vdots \\ r_m \end{pmatrix}\cdot(v_1 \cdots v_k) = \begin{pmatrix} r_1\cdot v_1 & r_1\cdot v_2 & \cdots & r_1\cdot v_k \\ \vdots & \vdots & & \vdots \\ r_m\cdot v_1 & r_m\cdot v_2 & & r_m v_k \end{pmatrix}$

$$\parallel$$

$$(Mv_1\ Mv_2 \cdots Mv_k)$$

and similarly for MB and $M(A|B)$

17. <u>Proof</u>: $B: 3\times 1$, $L_B: \mathbb{R}^1 \to \mathbb{R}^3$

$C: 1\times 3$. $L_C: \mathbb{R}^3 \to \mathbb{R}^1$

$\text{rank}(BC) = \text{rank}(L_{BC}) = \dim(R(L_{BC}))$.

$L_{BC} = L_B \cdot L_C \Rightarrow R(L_{BC}) = L_B(R(L_C)) \subseteq R(L_B)$

$\dim R(L_B) = \dim \mathbb{R}^1 - \dim N(L_B) \leq 1$.

$\Rightarrow \dim(R(L_{BC})) \leq \dim R(L_B) \leq 1$. ✓

Conversely, if A 3×3 has rank 1, by Corollary 1 to Thm 3.6, there exist invertible matrices P and Q of size 3×3 s.t.

$$PAQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (1\ 0\ 0).$$

$$\Rightarrow A = P^{-1} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \cdot (1\ 0\ 0)\, Q^{-1}$$

set $\underset{3\times 1}{B = P^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$ and $\underset{1\times 3}{C = (1\ 0\ 0)\, Q^{-1}}$

Then $A = B\cdot C$ as required. ∎