2.5 3(d) $\beta = \{x^2-x+1,\ x+1,\ x^2+1\} = \{v_1, v_2, v_3\}$

$\beta' = \{x^2+x+4,\ 4x^2-3x+2,\ 2x^2+3\}$

change of coordinate matrix $Q = [Id]_{\beta'}^{\beta}$. We need to express vectors in β' as linear combinations of β. Because it is easy to write v_i as a linear combination of standard basis $\{1, x, x^2\}$. We just need to write $\{1, x, x^2\}$ as linear combinations of v_i, which is not very difficult:

$$x = -(x^2-x+x) + (x^2+1) = -v_1 + v_3$$

$$1 = (x+1) - x = v_2 - (-v_1+v_3) = v_1 + v_2 - v_3$$

$$x^2 = (x^2+1) - 1 = v_3 - (v_1+v_2-v_3) = -v_1 - v_2 + 2v_3$$

So we can proceed:

$$x^2+x+4 = 4\cdot(v_1+v_2-v_3) + (-v_1+v_3) + (-v_1-v_2+2v_3)$$
$$= 2v_1 + 3v_2 - v_3$$

$$4x^2-3x+2 = 2\cdot(v_1+v_2-v_3) - 3\cdot(-v_1+v_3) + 4\cdot(-v_1-v_2+2v_3)$$
$$= v_1 - 2v_2 + 3v_3$$

$$2x^2+3 = 3\cdot(v_1+v_2-v_3) + 2\cdot(-v_1-v_2+2v_3)$$
$$= v_1 + v_2 + v_3$$

So we get $\underbrace{[Id]_{\beta'}^{\beta}}_{Q} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$.

There are different ways to get Q:

Let $\gamma = \{1, x, x^2\}$ be the standard basis. Then

$$Q = [Id]_{\beta'}^{\beta} = [Id]_{\gamma}^{\beta} \cdot [Id]_{\beta'}^{\gamma} = ([Id]_{\beta}^{\gamma})^{-1} \cdot [Id]_{\beta'}^{\gamma}$$

$[Id]_{\beta'}^{\gamma} = \begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix}$ can be easily written down

To calculate $[Id]_{\gamma}^{\beta} = ([Id]_{\beta}^{\gamma})^{-1}$, we can first write down:

$$[Id]_{\beta}^{\gamma} = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

and calculate its inverse by a standard process.

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

$$\downarrow$$

$$[Id]_{\gamma}^{\beta} = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix} \Leftarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 2 \end{array}\right)$$

So: $Q = \begin{pmatrix} 1 & -1 & -1 \\ 1 & 0 & -1 \\ -1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 4 & 2 & 3 \\ 1 & -3 & 0 \\ 1 & 4 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{pmatrix}$

5. $T: P_1(\mathbb{R}) \to P_1(\mathbb{R})$

$$T(p(x)) = p'(x).$$

$$\beta = \{1, x\}, \quad \beta' = \{1+x, 1-x\} \Rightarrow [Id]_{\beta'}^{\beta} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = Q$$

$$Q^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \frac{1}{1\cdot(-1)-1} \cdot \begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$[T]_\beta = \Big[\underbrace{[T(1)]_\beta}_{=[0]_\beta} \ \underbrace{[T(x)]_\beta}_{=[1]_\beta} \Big] = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$[T]_{\beta'} = Q^{-1} \cdot [T]_\beta \cdot Q = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$= \frac{1}{2} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

<u>Verify</u>: $T(1+x) = 1 = \frac{1}{2}\cdot(1+x) + \frac{1}{2}\cdot(1-x).$

$T(1-x) = -1 = -\frac{1}{2}\cdot(1+x) - \frac{1}{2}(1-x).$

8. $T: V \to V$ Prove $[T]_{\beta'}^{\gamma'} = P^{-1}[T]_{\beta}^{\gamma} Q$

<u>Proof</u>: $[T]_{\beta'}^{\gamma'} = [Id_V]_{\gamma}^{\gamma'} [T]_{\beta}^{\gamma} \cdot [Id_V]_{\beta'}^{\beta}$

$$= \left([Id_V]_{\gamma'}^{\gamma}\right)^{-1} [T]_{\beta}^{\gamma} \cdot [Id_V]_{\beta'}^{\beta}$$

$$= P^{-1} \cdot [T]_{\beta}^{\gamma} \cdot Q$$

where

$P = [Id_V]_{\gamma'}^{\gamma}$ is the matrix that changes γ'-coordinates into γ-coordinates

$Q = [Id_V]_{\beta'}^{\beta}$ – – – – – – – . β'-coordinates - - . β-coordinates.