HW4:

2.1.3: $T: \mathbb{R}^2 \to \mathbb{R}^3 \quad T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2)$

- $(a_1, a_2) \in N(T) \Leftrightarrow \begin{cases} a_1 + a_2 = 0 \\ 2a_1 - a_2 = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \end{cases} \Rightarrow N(T) = \{(0,0)\}$

  $\Rightarrow \text{nullity}(T) = 0.$

- $T(a_1, a_2) = (a_1 + a_2, 0, 2a_1 - a_2) = (a_1, 0, 2a_1) + (a_2, 0, -a_2) \quad \forall a_1, a_2 \in \mathbb{R}$

  $= a_1 \cdot (1, 0, 2) + a_2 \cdot (1, 0, -1).$

$\Rightarrow R(T) = \text{Span}\{(1,0,2), (1,0,-1)\}.$

$\{(1,0,2), (1,0,-1)\}$ is also linearly independent, so it is a basis for $R(T)$

$\Rightarrow \text{rank}(T) = 2.$

dimension equality: $\text{nullity}(T) + \text{rank}(T) = 0 + 2 = 2 = \dim \mathbb{R}^2.$

- $N(T) = \{0\} \Rightarrow T$ is one-to-one. (by Theorem 2.4)
- $(0,1,0) \notin R(T) \Rightarrow T$ is not onto.

2.1.9. (a) $T(0,0) = (1,0) \neq (0,0).$

(b) $T(c(a_1, a_2)) = (ca_1, c^2 a_2^2) \neq c(a_1, a_2^2) = c \cdot T(a_1, a_2)$

(c) $T(c(a_1, a_2)) = (\sin(ca_1), 0) \neq c(\sin(a_1), 0) = c \cdot T(a_1, a_2)$

(d) $T(c(a_1, a_2)) = (|c| \cdot |a_1|, ca_2) \neq c(|a_1|, a_2) = c \cdot T(a_1, a_2)$ if $c < 0.$

(e) $T(0,0) = (1,0) \neq (0,0)$

2.1.15. $T: P_2(\mathbb{R}) \to P_3(\mathbb{R})$ $\quad T(f(x)) = \int_0^x f(t)\,dt$

Prove that T is linear and one-to-one, but not onto.

<u>Proof</u>: • Prove T is linear: $\forall f_1, f_2 \in P_2(\mathbb{R}),\ \forall c \in \mathbb{R}$.

$$T(f_1+f_2) = \int_0^x (f_1(t)+f_2(t))\,dt = \underbrace{\int_0^x f_1(t)\,dt}_{T(f_1)} + \underbrace{\int_0^x f_2(t)\,dt}_{T(f_2)}$$

$$T(cf_1) = \int_0^x c\cdot f_1(t)\,dt = c\cdot\int_0^x f_1(t)\,dt = c\cdot T(f_1). \qquad \checkmark.$$

• T is one-to-one:

$$T(f_1) = T(f_2) \Leftrightarrow \int_0^x f_1(t)\,dt = \int_0^x f_2(t)\,dt$$

$$\xRightarrow{\text{taking derivative w.r.t. } x} \underbrace{\frac{d}{dx}\int_0^x f_1(t)\,dt}_{f_1(x)} = \underbrace{\frac{d}{dx}\int_0^x f_2(t)\,dt}_{f_2(x)} \qquad \checkmark.$$

• T is not onto: any $\underbrace{g}_{T(f) = \int_0^x f(t)\,dt} \in R(T)$ satisfies $g(0) = 0$.

so in particular $1 \notin R(T)$ and $R(T) \neq P_3(\mathbb{R})$.

2.1.28. Projection of V on W_1 along W_2

$$V = W_1 \oplus W_2 \longrightarrow V$$
$$x = x_1 + x_2 \longmapsto x_1$$

(a) W a subspace of a finite-dim. subspace of V.

Choose a basis $\beta = \{v_1, \cdots, v_m\}$ of W.

By [Corollary 2, pg. 48], β can be extended to a basis

$$\gamma = \{v_1, \cdots, v_m, u_1, \cdots, u_k\} \text{ of } V.$$

Define $W' = \mathrm{Span}\{u_1, \cdots, u_k\}$. Then $V = W \oplus W'$

Define a function: $T: V \to V$ s.t. $T(v_i) = v_i,\ i = 1, \cdots, m$
$T(u_j) = 0,\ j = 1, \cdots, k$.

Then T is uniquely defined (determined by its values on a basis)

and is a projection on W along W'.

(b) $V = \mathbb{R}^2$, $W = \left\{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R}\right\} = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$

Let W' be any 1-dim subspace spanned by $\underset{\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}}{\overset{u}{\shortparallel}} \in \mathbb{R}^2$ s.t. $v \notin W$.

Then $V = W \oplus W'$ because $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}\right\}$ is a basis for $V = \mathbb{R}^2$

Then we get the associated projection depending on $W' = \mathrm{span}\{u\}$.

Different choice of W' gives different projections.

For example:

- $u = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ gives the standard projection: $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$.
- $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$: $\begin{pmatrix} x \\ y \end{pmatrix} = (x-y) \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 1 \\ 1 \end{pmatrix} \in W \oplus W'$.

  So this u gives the projection: $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x-y \\ 0 \end{pmatrix}$

2.2. 2(a) (β and γ are standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$
$\Rightarrow [v]_\beta = v$ for any $v \in \mathbb{R}^n$. $[w]_\gamma = w$ $\forall w \in \mathbb{R}^m$.)

$T: \mathbb{R}^2 \to \mathbb{R}^3$ $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

$T(1,0) = (2,3,1)$, $T(0,1) = (-1, 4, 0)$

$\Rightarrow [T]_\beta^\gamma = \begin{pmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{pmatrix}$

(g) $T: \mathbb{R}^n \to \mathbb{R}$ $T(a_1, a_2, \cdots, a_n) = a_1 + a_n$.

$T(e_1) = T(1, 0, \cdots, 0) = 1$

$T(e_2) = T(0, 1, \cdots, 0) = 0$

$\cdots$

$T(e_n) = T(0, 0, \cdots, 1) = 1$

$\Rightarrow [T]_\beta^\gamma = (1, 0, \cdots, 0, 1)$ $1 \times n$ matrix.

4. $T: M_{2\times2}(\mathbb{R}) \to P_2(\mathbb{R})$ $\quad T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a+b) + (2d)x + bx^2$.

$\beta = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$, $\gamma = \{1, x, x^2\}$

$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = 1 + 0\cdot x + 0\cdot x^2$, $T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = 1 + 0\cdot x + 1\cdot x^2$

$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = 0 + 0\cdot x + 0\cdot x^2$, $T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0 + 2\cdot x + 0\cdot x^2$

$$\Rightarrow [T]_\beta^\gamma = \left(\begin{array}{c|c|c|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \end{array}\right)$$

8. Prove $T: V^n \to F^n$, $T(x) = [x]_\beta$ is linear

Pf: Assume $\beta = \{v_1, \ldots, v_n\}$

- Prove T preserves addition: Let $x, y \in V$

$x = a_1 v_1 + \cdots + a_n v_n \iff [x]_\beta = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$y = b_1 v_1 + \cdots + b_n v_n \iff [y]_\beta = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$

$\Rightarrow x + y = (a_1 + b_1) v_1 + \cdots + (a_n + b_n) v_n$

$\Rightarrow [x+y]_\beta = \begin{pmatrix} a_1 + b_1 \\ \vdots \\ a_n + b_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} + \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = [x]_\beta + [y]_\beta \Rightarrow T(x+y) = T(x) + T(y)$

- Prove T preserves scalar multiplication:

$x = a_1 v_1 + \cdots + a_n v_n$, $c \in F$ $\Rightarrow c\cdot x = (ca_1)\cdot v_1 + \cdots + (ca_n)\cdot v_n$

$T(c\cdot x) = [cx]_\beta = \begin{pmatrix} ca_1 \\ \vdots \\ ca_n \end{pmatrix} = c\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} = c[x]_\beta = c\cdot T(x)$ $\quad \forall x \in V$. ∎

10. $\beta = \{v_1, v_2, \cdots, v_n\}$, define $v_0 = 0$.

$$T(v_j) = v_j + v_{j-1} = v_{j-1} + v_j$$

$$\Rightarrow \quad T(v_1) = v_0 + v_1 = v_1$$

$$T(v_2) = v_1 + v_2$$

$$\cdots$$

$$T(v_n) = v_{n-1} + v_n$$

$$\Rightarrow [T]_\beta = \begin{array}{c} \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & & \uparrow \\ [T(v_1)]_\beta & [T(v_2)]_\beta & & [T(v_n)]_\beta \end{array} \end{array}$$

$$= \begin{pmatrix} 1 & 1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}$$