HW 3:

1.5 2(e): Determine linearly dependent/independent:

(e) $\{(1,-1,2), (1,-2,1), (1,1,4)\}$ in $\mathbb{R}^3$

Determine whether it is linearly independent: Find relations $\Updownarrow$ representations of 0.

$$\begin{pmatrix}0\\0\\0\end{pmatrix}=a_1\begin{pmatrix}1\\-1\\2\end{pmatrix}+a_2\begin{pmatrix}1\\-2\\1\end{pmatrix}+a_3\begin{pmatrix}1\\1\\4\end{pmatrix}=\begin{pmatrix}1&1&1\\-1&-2&1\\2&1&4\end{pmatrix}\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$$

$$\begin{pmatrix}1&1&1\\-1&-2&1\\2&1&4\end{pmatrix}\xrightarrow[①\cdot(-2)+③]{①+②}\begin{pmatrix}1&1&1\\0&-1&2\\0&-1&2\end{pmatrix}\longrightarrow\begin{pmatrix}1&1&1\\0&1&-2\\0&0&0\end{pmatrix}\to\begin{pmatrix}1&0&3\\0&1&-2\\0&0&0\end{pmatrix}\Leftrightarrow\begin{cases}a_1=-3a_3\\a_2=2a_3\end{cases}$$

there are nonzero solutions of $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}$ (because there is a free variable $a_3$)

$\Rightarrow$ there are nontrivial rep. of 0 $\Rightarrow$ linearly dependent.

( verify: $\begin{pmatrix}a_1\\a_2\\a_3\end{pmatrix}=\begin{pmatrix}-3\\2\\1\end{pmatrix}\to -3\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}+2\begin{pmatrix}1\\-2\\1\end{pmatrix}+1\cdot\begin{pmatrix}1\\1\\4\end{pmatrix}=\begin{pmatrix}-3+2+1\\3-4+1\\-6+2+4\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$. )

1.5.9, $u\neq v\in V$. Prove: $\{u,v\}$ is linearly depend. $\underset{\text{iff}}{\Leftrightarrow}$ $u$ or $v$ is a multiple of the other

Proof: "If" If $u=c\cdot v$, then $(-1)\cdot u+c\cdot v=0\Rightarrow$ linearly dependent.
If $v=c\cdot u$. then $cu+(-1)\cdot v=0\Rightarrow$

"only if" if $\{u,v\}$ is linearly dependent, then $\exists\,(a,b)\neq(0,0)$ s.t. $au+bv=0$.

two cases:
- $a\neq0$: then $u=-a^{-1}b\cdot v$, $u$ is a multiple of $v$
- $b\neq0$: then $v=-b^{-1}a\cdot u$. $v$ is a multiple of $u$ ∎

1.5.15 $S = \{u_1, u_2, \cdots, u_n\}$.

Prove $S$ is linearly dependent iff $u_1 = 0$ or $u_{k+1} \in \mathrm{Span}(\{u_1, \cdots, u_k\})$ for some $k$, $1 \le k < n$.

Proof: "if" if $u_1 = 0$, then $S$ is linearly dependent: $0 = 1 \cdot u_1$

if $u_{k+1} \in \mathrm{Span}(\{u_1, \cdots, u_k\})$, then $u_{k+1} = c_1 u_1 + \cdots + c_k u_k$, $1 \le k < n$.

$\Rightarrow (-1) \cdot u_{k+1} + a_1 \cdot u_1 + \cdots + c_k \cdot u_k = 0 \Rightarrow S$ is linearly dependent

"only if": Assume $S$ is linearly dependent. Then $\exists a_1, \cdots, a_n \in F$ not all zero.

s.t. $0 = a_1 \cdot u_1 + a_2 u_2 + \cdots + a_n \cdot u_n$.

Let $m$ be the maximum of $i \in \{1, \cdots, n\}$ such that $a_i \neq 0$.

If $m = 1$. then $0 = a_1 u_1 + 0 \cdot u_2 + \cdots + 0 \cdot u_n = a_1 u_1 \Rightarrow u_1 = 0$

If $m > 1$, then $0 = a_1 u_1 + a_2 u_2 + \cdots + a_m \cdot u_m + 0 \cdot u_{m+1} + \cdots + 0 \cdot u_n$

$\Rightarrow u_m = -a_m^{-1} a_1 u_1 - a_m^{-1} a_2 u_2 - \cdots - a_m^{-1} a_{m-1} u_{m-1}$

$\Rightarrow \underset{\substack{\| \\ u_{k+1}}}{u_m} \in \underset{\substack{\| \\ \mathrm{Span}\{u_1, \cdots, u_k\}}}{\mathrm{Span}\{u_1, \cdots, u_{m-1}\}}$.   let $k = m-1 \in \{1, \cdots, n-1\}$ ∎

1.6. 3(a): Determine whether the subset is a basis for $P_2(\mathbb{R})$.

$\{-1 - x + 2x^2,\ 2 + x - 2x^2,\ 1 - 2x + 4x^2\}$

Just need to determine if it is linearly dependent or not.

Find representation of 0 polynomial:

$0 = a_1(-1 - x + 2x^2) + a_2(2 + x - 2x^2) + a_3(1 - 2x + 4x^2)$

$$\Leftrightarrow \begin{cases} -a_1 + 2a_2 + a_3 = 0 \\ -a_1 + a_2 - 2a_3 = 0 \\ 2a_1 - 2a_2 + 4a_3 = 0 \end{cases}$$

$$\Leftrightarrow \begin{pmatrix} -1 & 2 & 1 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 2 & 1 \\ -1 & 1 & -2 \\ 2 & -2 & 4 \end{pmatrix} \xrightarrow[\textcircled{1}\cdot 2+\textcircled{3}]{\textcircled{1}\cdot(-1)+\textcircled{2}} \begin{pmatrix} -1 & 2 & 1 \\ 0 & -1 & -3 \\ 0 & 2 & 6 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -2 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{cases} a_1 = -5a_3 \\ a_2 = -3a_3 \end{cases} \Rightarrow \exists \text{ nonzero solutions } \left(\text{when } a_3 \neq 0,\ \text{e.g. } \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix}\right)$$

So the set is not linearly independent and is <u>not</u> a basis.

3.(b) $\{1+2x+x^2,\ 3+x^2,\ x+x^2\}$. Similar method:

$$a_1\cdot(1+2x+x^2) + a_2\cdot(3+x^2) + a_3\cdot(x+x^2) = 0 \Leftrightarrow \begin{cases} a_1+3a_2=0 \\ 2a_1+a_3=0 \\ a_1+a_2+a_3=0. \end{cases}$$

$$\begin{pmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & -6 & 1 \\ 0 & -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow \text{no free variables}$$

$$\Downarrow$$

only zero solutions for $\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$

So the set is linearly independent.

Because it consists of $\underset{\dim P_2(\mathbb{R})}{\underset{\shortparallel}{3}}$ polynomials, by [Corollary 2(b), pg. 48], it is a basis for $P_2(\mathbb{R})$.

15. $W = \{ A \in M_{n\times n}(F) : \text{trace}(A) = 0 \}$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \in W \iff a_{11} + a_{22} + \cdots + a_{nn} = 0$$

$$\Updownarrow$$

$$a_{nn} = -a_{11} - a_{22} - \cdots - a_{n-1\,n-1}$$

$$\Rightarrow A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & & \cdots & \\ a_{n1} & a_{n2} & \cdots & -a_{11} - a_{22} - \cdots - a_{nn} \end{pmatrix}$$

basis vectors: $E^{ij} = \begin{array}{c} \begin{pmatrix} 0 & & 0 & 0 \\ \cdots & \cdots & 1 & \cdots \\ 0 & 0 & 0 & 0 \end{pmatrix} i \\ j \end{array}$ $\quad i \neq j$

$$E^{11} - E^{nn} = \begin{pmatrix} 1 & & & 0 \\ & 0 & & \\ & & \ddots & \\ 0 & & 0 & -1 \end{pmatrix}$$

$$E^{22} - E^{nn} = \begin{pmatrix} 0 & & & 0 \\ & 1 & & \\ & & 0 & \\ & & & \ddots \\ 0 & & & -1 \end{pmatrix}$$

$$\cdots$$

$$E^{n-1,n-1} - E^{nn} = \begin{pmatrix} 0 & & & 0 \\ & \ddots & & \\ & & 0 & \\ & & & 1 \\ 0 & & & & -1 \end{pmatrix}$$

$\dim = n(n-1) + (n-1) = n^2 - 1$

21. Prove that a vector space is infinite dimensional iff it contains an infinite linearly independent subset.

<u>Proof</u>: A vector space is called infinite dimensional if it has no basis consisting of finite number of vectors. ( Definitions on page 47 )

"if" Assume $V$ contains an infinite linearly independent subset.

If $V$ contains a basis $\beta$ with $\#\beta = \dim V < +\infty$,

then by the Replacement Theorem, any linearly independent subset has at most $\#\beta = \dim V$ vectors (since $V$ is generated by $\beta$).

This contradicts the assumption. So $V$ must be infinite dimensional.

"only if" $V$ is infinite dimensional. We can construct an infinite linearly independent subset in the following way by induction:

step 1: Choose a nonzero $v_1 \in V$, set $S_1 = \{v_1\}$.

step $n \Rightarrow$ step $n+1$: Assume we have a linearly independent subset ($n \in \mathbb{N} = \{1, 2, 3, \cdots\}$) $S_n = \{v_1, \cdots, v_n\}$ of size $n$.

Then $S_n$ can not span $V$ because otherwise $S_n$ contains a basis of $V$ by [Theorem 1.9, P.g. 45]. So there exists $v_{n+1} \in V$ s.t. $v_{n+1} \notin \mathrm{Span}(S_n)$. By [Theorem 1.7, Pg 41], the subset

$S_{n+1} = S_n \cup \{v_{n+1}\}$ is linearly independent.

So in this way, we can construct an infinite linearly independent subset $\{v_i : i \in \mathbb{N}\}$ ∎

26. subspace of $P_n(\mathbb{R})$ $\quad W=\{f\in P_n(\mathbb{R}) : f(a)=0\}$.

$P_n(\mathbb{R}) \ni f = b_0 + b_1 \cdot x + b_2 x^2 + \cdots + b_n \cdot x^n$ with $b_0, \ldots, b_n \in \mathbb{R}$

is contained in $W$ if and only if

$$b_0 + b_1 \cdot a + b_2 \cdot a^2 + \cdots + b_n \cdot a^n = 0 \Leftrightarrow b_0 = -b_1 a - b_2 a^2 - \cdots - b_n \cdot a^n$$

$$\Leftrightarrow f = (-b_1 a - b_2 \cdot a^2 - \cdots - b_n a^n) + b_1 \cdot x + \cdots + b_n \cdot x^n$$

$$= b_1 \cdot (-a+x) + b_2 \cdot (-a^2 + x^2) + \cdots + b_n \cdot (-a^n + x^n)$$

so $W = \operatorname{Span} \beta$ where

$$\beta = \{-a+x, -a^2+x^2, \ldots, -a^n+x^n\}$$

It is easy to show that $\beta$ is linearly independent (by using the fact the polynomials in $\beta$ are of different degrees)

so $\beta$ is a basis for $W$ and $\dim W = n$

$$n = (n+1) - 1 = \dim P_n(\mathbb{R}) - 1.$$