HW 2 Sol:

1.3.18: Prove a subset W of a vector space V is a subspace of V iff $0 \in W$ and $ax+y \in W$, whenever $a \in F$ and $x, y \in W$.

Pf: W is a subspace of V if the following conditions are satisfied

(1). $x, y \in W \Rightarrow x+y \in W$

(2). $x \in W, a \in F \Rightarrow a \cdot x \in W$

(3). $0 \in W$.

These conditions are equivalent to the two conditions:

(i) $x, y \in W, a \in F \Rightarrow ax+y \in W$

(ii) $0 \in W$.

More precisely: $(1)+(2) \Leftrightarrow (i)$, $(3) \Leftrightarrow (ii)$.

$$\left(\begin{array}{ccc} (1) & \xLeftarrow{\text{set } a=1} & (i) \\ (2) & \swarrow_{\text{set } y=0} & \end{array} \right)$$

$\blacksquare$

1.3.19: W_1, W_2 subspaces of V. Prove that $W_1 \cup W_2$ is a subspace of V iff $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$.

Pf: "if" = sufficient condition = $\Leftarrow$

If $W_1 \subseteq W_2$, then $W_1 + W_2 = W_2$ is a subspace

If $W_2 \subseteq W_1$, then $W_2 + W_1 = W_1$ is a subspace

"only if" $\Leftrightarrow$ necessary condition:

Assume that $W_1 \cup W_2$ is a subspace of V.

Case 1: If $W_1 \subseteq W_2$, then we are done.

Case 2: If $W_1 \nsubseteq W_2$, then $\exists\ x \in W_1 \setminus W_2$,
($x \in W_1$, but $x \notin W_2$).

Pick any $y \in W_2$, then $x, y \in W_1 \cup W_2$

and $x+y \in W_1 \cup W_2$ (because $W_1 \cup W_2$ is assumed to be a subspace)

So $x+y \in W_1$ or $x+y \in W_2$.

(i) If $x+y = z \in W_1$, then $y = z - x \in W_1$ (because $z \in W_1$ and $x \in W_1$)

(ii) If $x+y = z \in W_2$, then $x = z - y \in W_2$ (because $z \in W_2$ and $y \in W_2$).

but this contradicts the condition that $x \notin W_2$.

So case (ii) can not happen.

So we must have condition (i) which shows that $y \in W_1$ for any $y \in W_2$.

So we must have $W_2 \subseteq W_1$ in case 2. ∎

1.3.23: W_1, W_2 are subspaces of V

(a) Prove that $W_1 + W_2$ is a subspace of V that contains both W_1 and W_2.

(b) Prove that any subspace of V that contains both W_1 and W_2 must also contain $W_1 + W_2$.

Pf: (a) $W_1 + W_2$ is closed under vector addition and scalar multiplication.

(b). If a subspace W_3 contains W_1 and W_2,

then $W_1 \subseteq W_3$ and $W_2 \subseteq W_3$

This implies $W_1 + W_2 \subseteq W_3 + W_3 = W_3$. 𝔅

1.4.4(a): In $P_3(\mathbb{R})$, determine whether x^3-3x+5 is a linear combination of x^3+2x^2-x+1 and x^3+3x^2-1.

Sol: Determine if there are $a, b \in \mathbb{R}$ that satisfy:

$$x^3-3x+5 = a\cdot(x^3+2x^2-x+1)+b\cdot(x^3+3x^2-1).$$

This is equivalent to the linear system:

$$\begin{cases} 1 = a+b & ① \\ 0 = 2a+3b & ② \\ -3 = -a & ③ \\ 5 = a-b & ④ \end{cases}$$

$$③ \Rightarrow a=3 \xRightarrow{④} b = a-5 = -2$$

Verify ②: $2a+3b = 2\cdot 3 + 3\cdot(-2) = 0$ ✓

① : $a+b = 3+(-2) = 1.$ ✓

So there is a solution $(a,b) = (3,2)$.

So x^3-3x+5 is a linear combination of the other the two polynomials

$\underline{1.4.5(g)}$: Determine whether $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ is in $\overbrace{\operatorname{Span}\left\{\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right\}}^{S}$

$\underline{\text{Sol}}$: Try to find $a, b, c \in \mathbb{R}$ s.t.

$$\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix} = \underbrace{a\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}}_{= \begin{pmatrix} a+c & b+c \\ -a & b \end{pmatrix}}.$$

$$\Longleftrightarrow \begin{cases} 1 = a + c & ① \\ 2 = b + c & ② \\ -3 = -a & ③ \\ 4 = b & ④ \end{cases}$$

$$③,④ \Longrightarrow a = 3,\ b = 4 \overset{②}{\Longrightarrow} c = 2 - b = -2$$

$$\overset{①}{\Longrightarrow} c = 1 - a = -2 \ \checkmark$$

So there is a solution: $(a, b, c) = (3, 4, -2)$.

and $\begin{pmatrix} 1 & 2 \\ -3 & 4 \end{pmatrix}$ is in $\operatorname{span}(S)$.

<u>1.4.12</u>: Prove that a subset W of a vector space V is a subspace iff $\mathrm{Span}(W) = W$.

<u>Pf</u>: "if" $W = \mathrm{span}(W)$ is a subspace because span of any (non-empty) subset is a subspace.

"only if" If W is a subspace, then any linear combination of vectors in W is contained in W. So $\mathrm{span}(W) \subseteq W$.

On the other hand, we always have $W \subseteq \mathrm{Span}(W)$, so we conclude that $\mathrm{span}(W) = W$.

$\underline{1.4.14}$: show that $\text{Span}(S_1 \cup S_2) = \text{Span}(S_1) + \text{Span}(S_2)$.

$\underline{\text{Pf}}$: 1. show: $\text{span}(S_1 \cup S_2) \subseteq \text{Span}(S_1) + \text{Span}(S_2)$.

Pick any $v \in \text{Span}(S_1 \cup S_2)$, then there ~~exist~~ exist

$u_1, \dots, u_n \in S_1 \cup S_2$, $a_1, \dots, a_2 \in F$ s.t. $v = a_1 u_1 + \dots + a_n u_n$

By re-arranging the vectors, we can assume that

$u_1, \dots, u_m \in S_1$ and $u_{m+1}, \dots, u_m \in S_2$

then $v = \underbrace{(a_1 u_1 + \dots + a_m u_m)}_{\text{Span}(S_1)} + \underbrace{(a_{m+1} u_{m+1} + \dots + a_n u_n)}_{\text{span}(S_2)}$

So $v \in \text{Span}(S_1) + \text{Span}(S_2)$.

2. show that $\text{span}(S_1) + \text{Span}(S_2) \subseteq \text{Span}(S_1 \cup S_2)$.

This is because: for $i = 1, 2$

$S_i \subseteq S_1 \cup S_2 \Rightarrow \text{Span}(S_i) \subseteq \text{Span}(S_1 \cup S_2)$

$\Rightarrow \text{Span}(S_1) + \text{Span}(S_2) \subseteq \underbrace{\text{Span}(S_1 \cup S_2) + \text{Span}(S_1 \cup S_2)}_{= \text{Span}(S_1 \cup S_2)}$ $\blacksquare$